B-Splines as Finite Elements

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Joint work with
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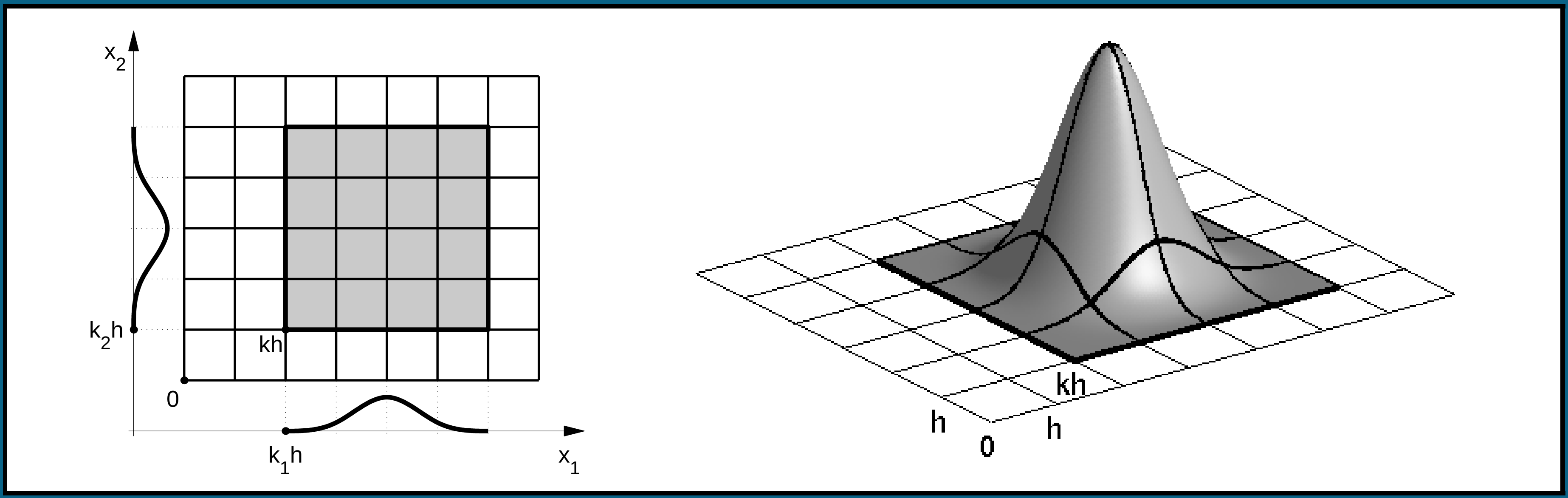
WCCM, July 17, 2006

Splines: The d-variate B-splines of order n on the uniform grid $h\mathbb{Z}^d$ are denoted $b_k^n,\ k \in \mathbb{Z}^d$.



If the domain is restricted to $\Omega \subset \mathbb{R}^d$, use only

$$B^n := \{b_k^n : \operatorname{supp} b_k^n \cap \Omega \neq \emptyset\}.$$

A spline is a linear combination of B-splines with control points,

$$s = \sum_k b_k^n p_k = B^n P.$$

Main features:

- high approximation order, $\|u - s_h\| = O(h^n)$
- few dof (one per grid cell)
- mesh generation is trivial
- local support yields small band widths
- local refinement
- efficient algorithms and data structures

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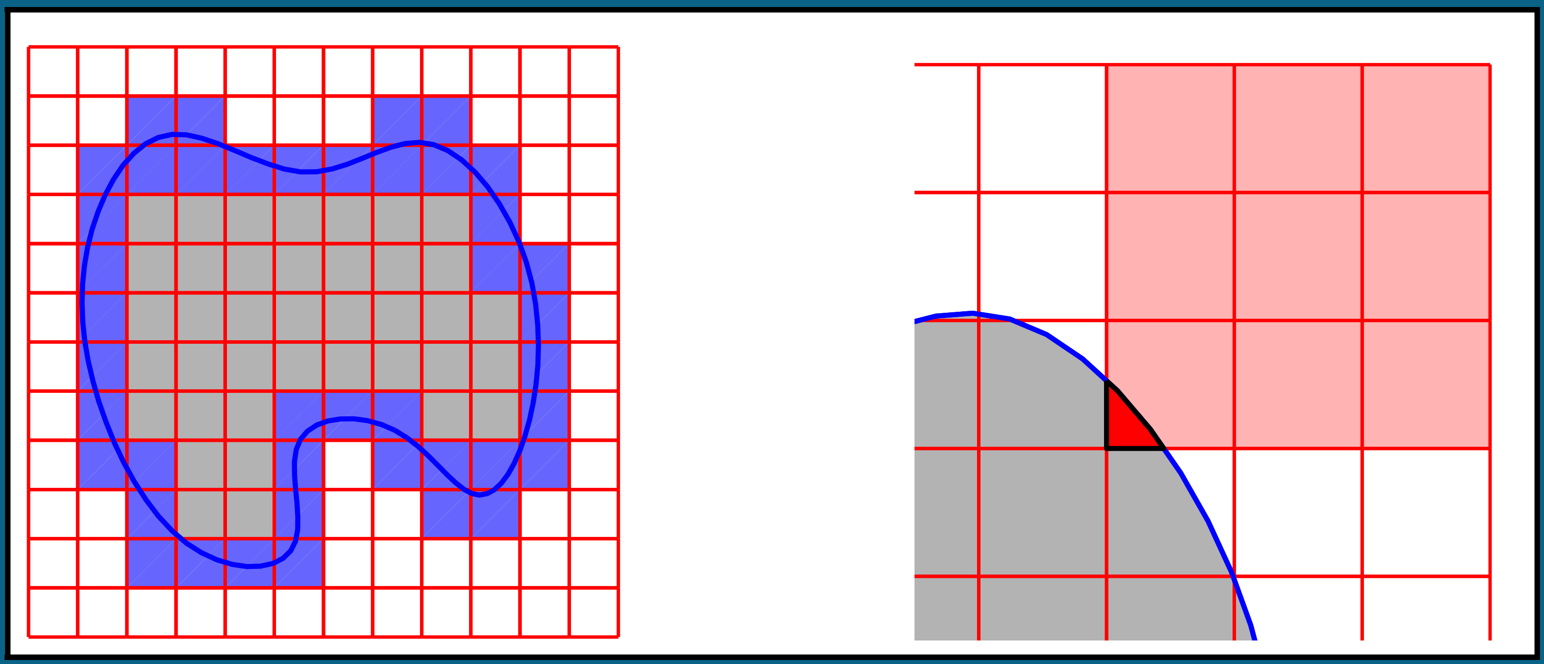
Why are B-splines not commonly used for FE-approximations?

Problems:

- Boundary conditions: If a spline is forced to be zero at the boundary of Ω, then it vanishes on all intersecting grid cells (in general). This implies a complete loss of approximation power.
- Condition number: B-splines with small support in Ω may lead to excessively large condition numbers. Leaving out outer B-splines reduces approximation power.



Extension: Partition the set K of relevant indices as follows:

The inner B-splines with indices

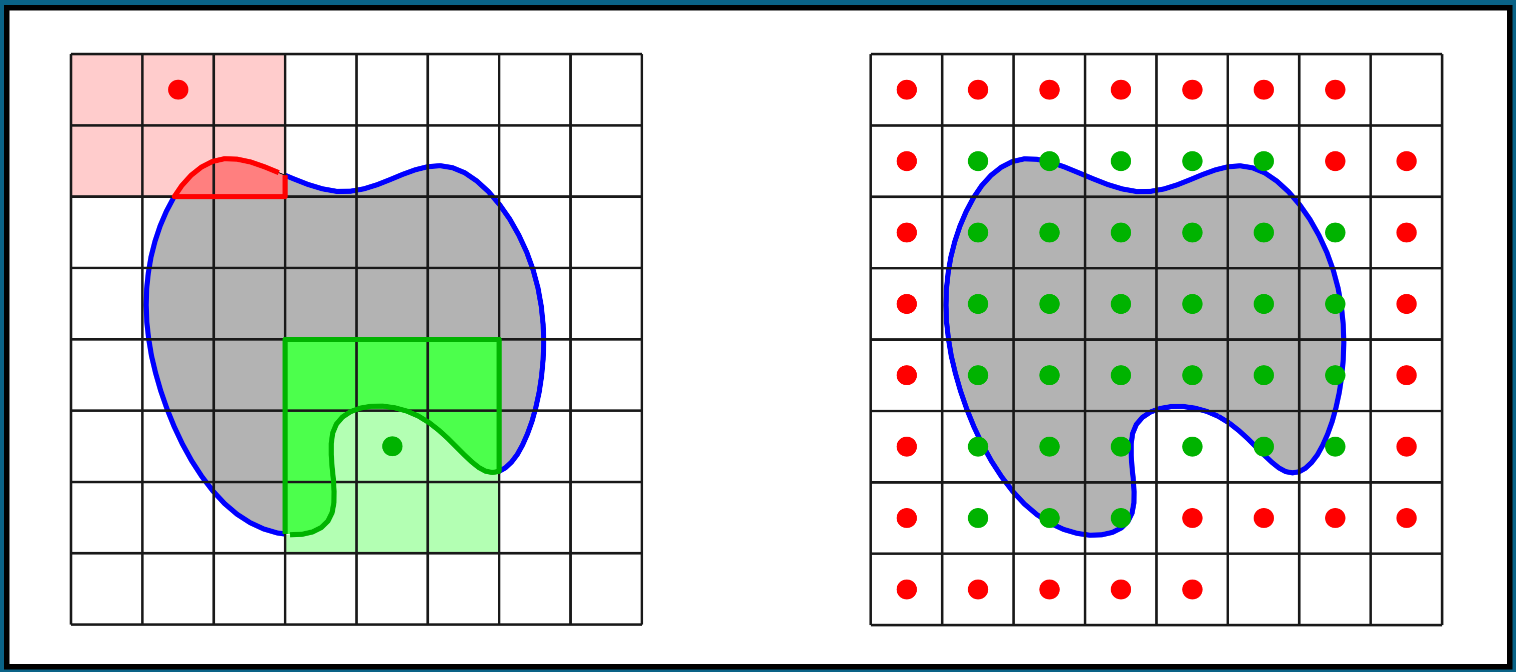
$$I \subset K$$

have at least one grid cell in their support contained in Ω.

The outer B-splines with indices

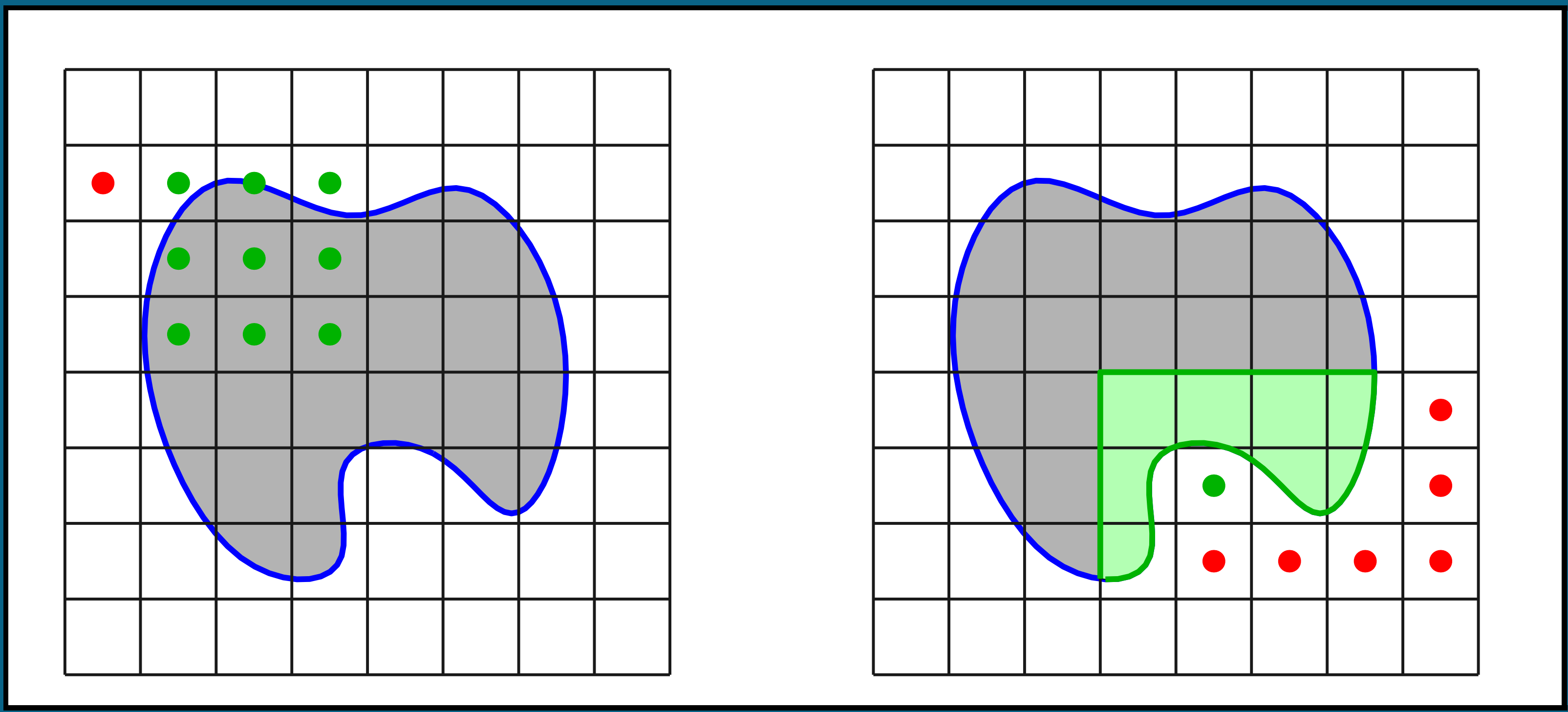
$$J = K \backslash I$$

have no grid cell in their support contained in Ω .



Extension: In order to stabilize the basis, the outer B-splines are no longer considered to be independent, but coupled with inner B-splines,

$$B_i = b_i + \sum_{j \in J} e_{i,j} b_j, \quad i \in I.$$

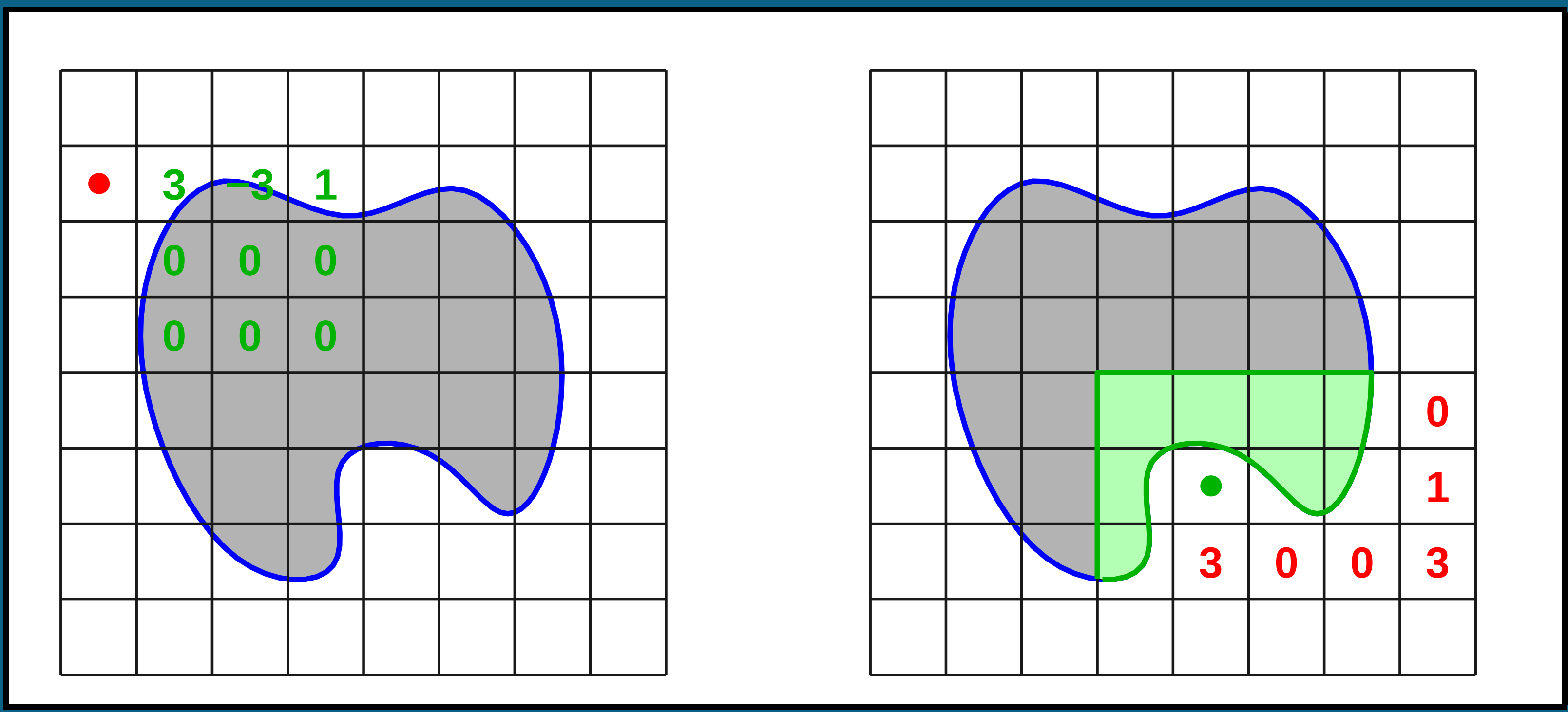


Extension:

Choose coefficients $e_{i,j}$ in such a way that all polynomials of order n remain in the span of the B_i using Marsden's identity,

$$\sum_{k \in K} p(k) b_k \in \mathbb{P}_n(\Omega) \quad \text{iff} \quad p \in \mathbb{P}_n(K).$$

and Lagrange interpolation of neighboring inner control points.



Weighting:

The incorporation of zero boundary conditions is amazingly simple. Let $w : \Omega \to \mathbb{R}_0^+$ be a smooth function equivalent to the boundary distance, i.e.

$$\frac{w(x)}{\operatorname{dist}(x, \partial\Omega)} \preceq 1, \quad \frac{\operatorname{dist}(x, \partial\Omega)}{w(x)} \preceq 1,$$

and in particular

$$w = 0 \quad \text{exactly on } \partial\Omega.$$

Multiplying the extended B-splines B_i by the weight function w yields a basis which satisfies the boundary condition.

Definition:

The weighted extended B-splines B_i (web-splines) are defined by

$$B_i = \frac{w}{w(x_i)}\Big(b_i + \sum_{j\in J(i)} e_{i,j} b_j\Big), \quad i \in I,$$

where $x(i)$ is the center of a grid cell in $\operatorname{supp} b_i \cap \Omega$.

The web-splines span the web-space

$$\mathbb{B} := \operatorname{span}\{B_i : i \in I\}.$$

Theorem: *The web-basis has essentially the same stability properties as the standard B-splines basis,*

$$\Big\|\sum_{i\in I} a_i B_i\Big\|_0 \sim \|A\| \;, \quad \Big\|\sum_{i\in I} a_i B_i\Big\|_r \preceq h^{-r}\, \|A\|.$$

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$$\operatorname{cond} G_h \preceq h^{-2}.$$

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Theorem: *The condition number of the Galerkin matrix is bounded by*

$$\operatorname{cond} G_h \preceq h^{-2}.$$

Theorem: *web-splines have essentially the same (optimal) approximation properties as standard B-splines. More precisely, for* $u \in H_0^1 \cap H^k$ *there exists a web-spline* s_h *such that for all* $k \leq n$

$$\|u - s_h\|_r \preceq \|u\|_k \, h^{k-r}.$$

Example 1: Helmholtz equation

Compute many (several thousand) eigenvalues of the Laplacian in order to understand quantum mechanics of small particles,

$$\Delta u + \lambda u = 0 \quad \textit{in } \Omega, \quad u = 0 \quad \textit{on } \partial\Omega.$$

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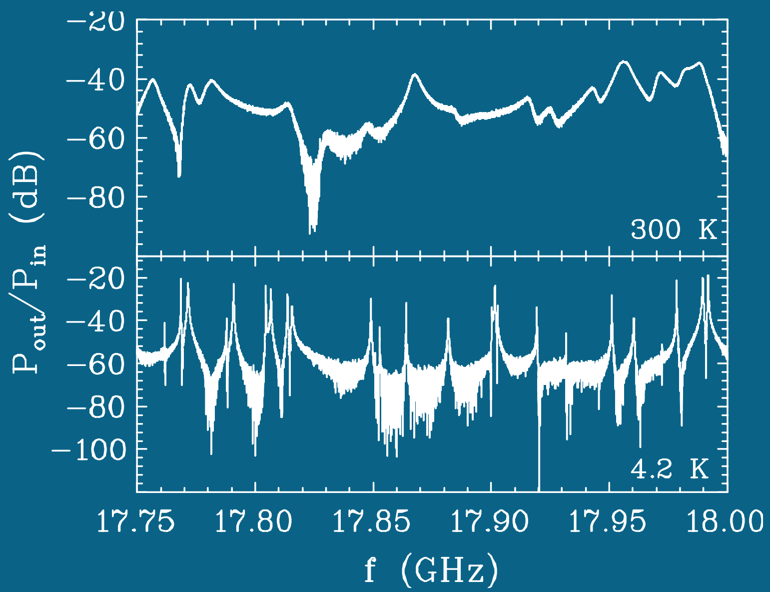

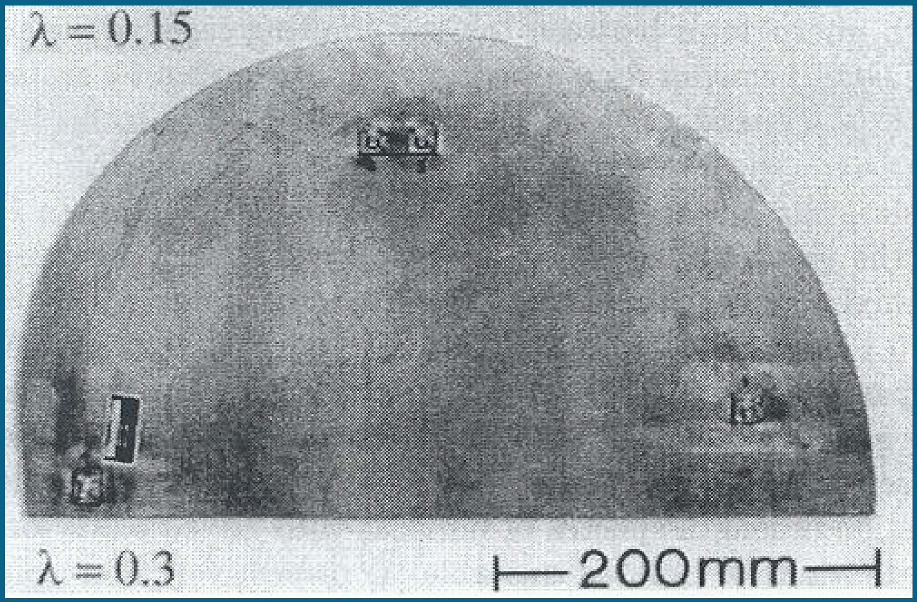

The FE-discretization leads to a generalized eigenvalue problem,

$$Av = \lambda Bv.$$

	# EV	CPU time (h)
standard technique	200	100

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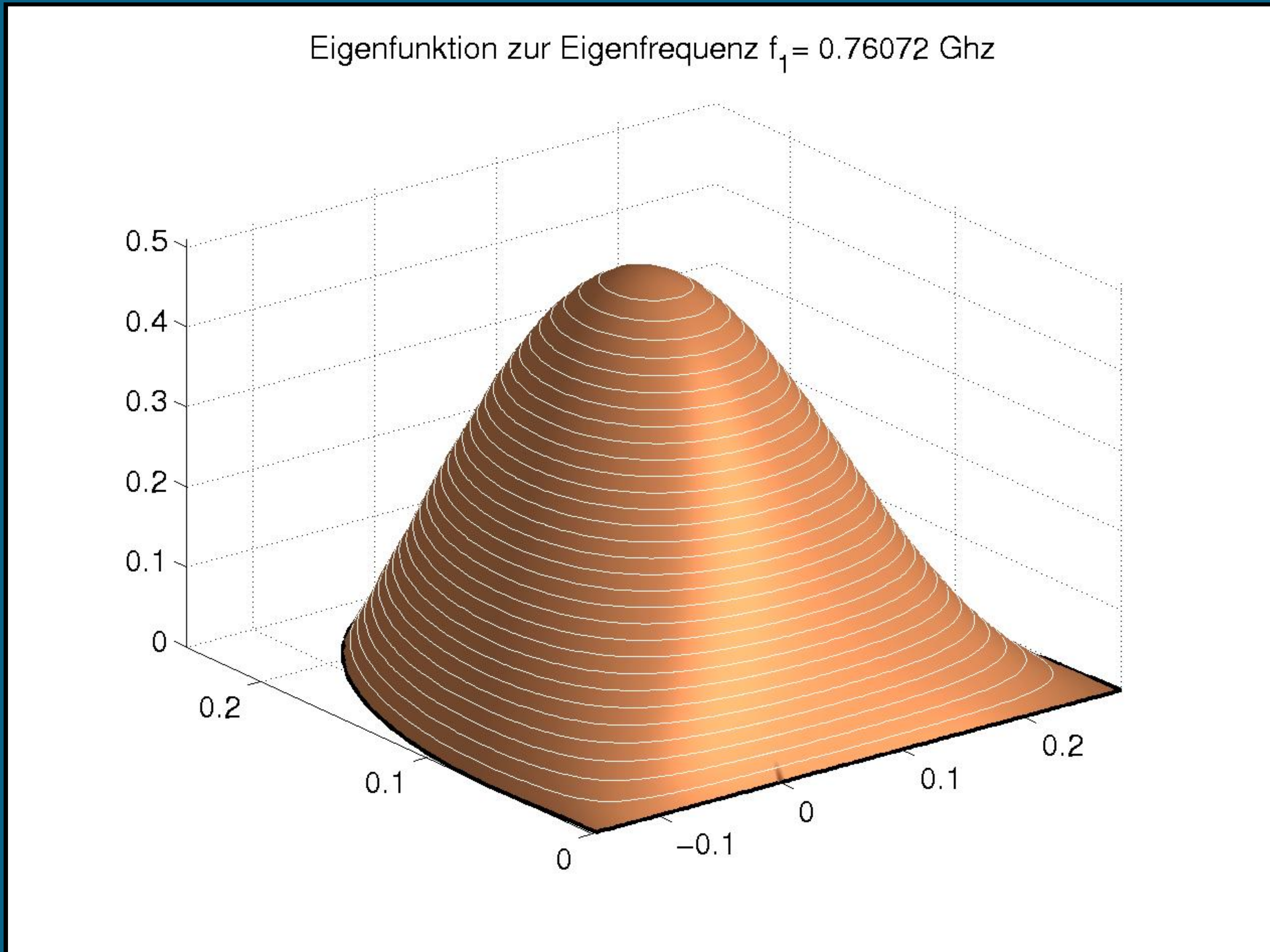
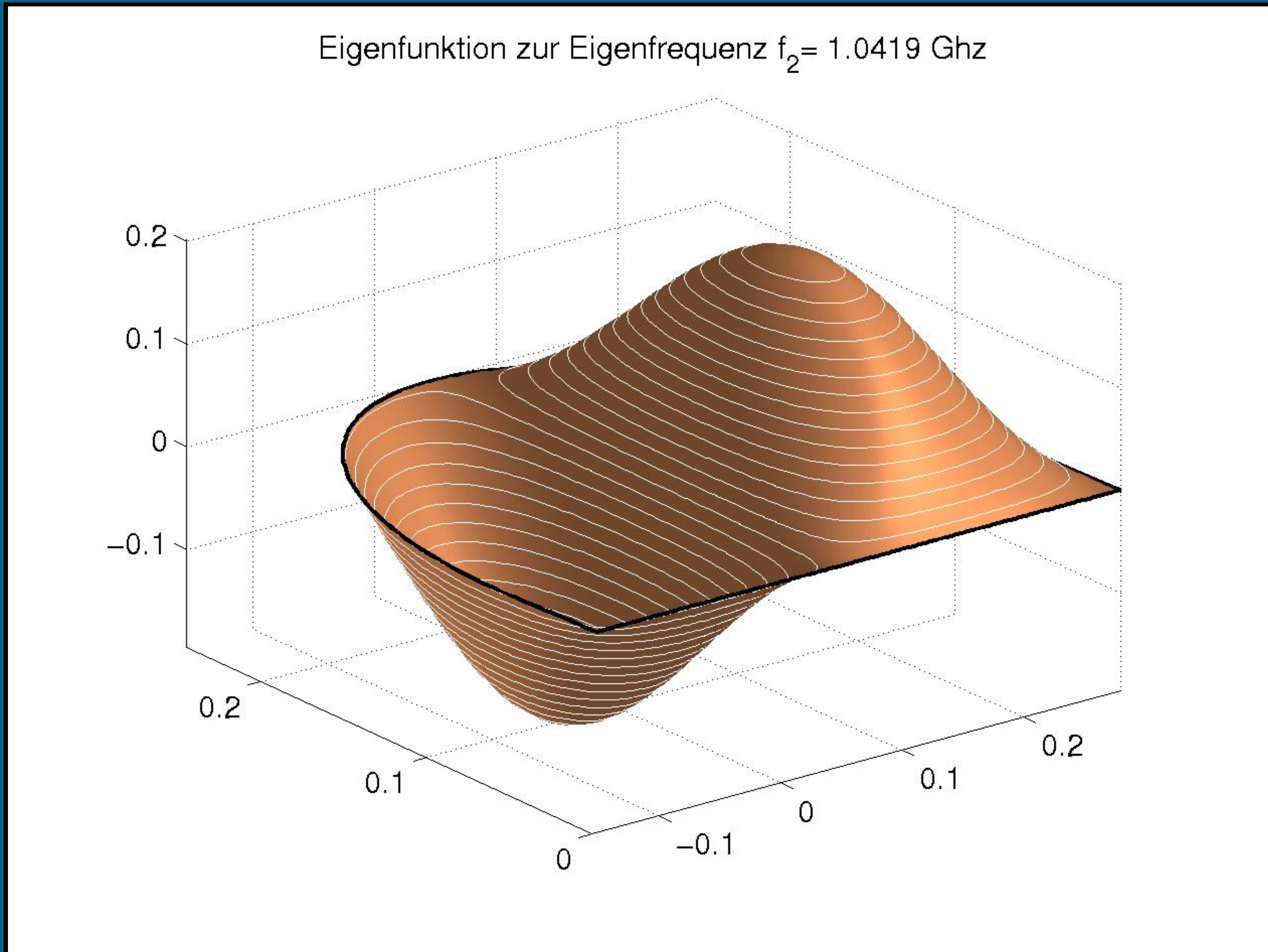
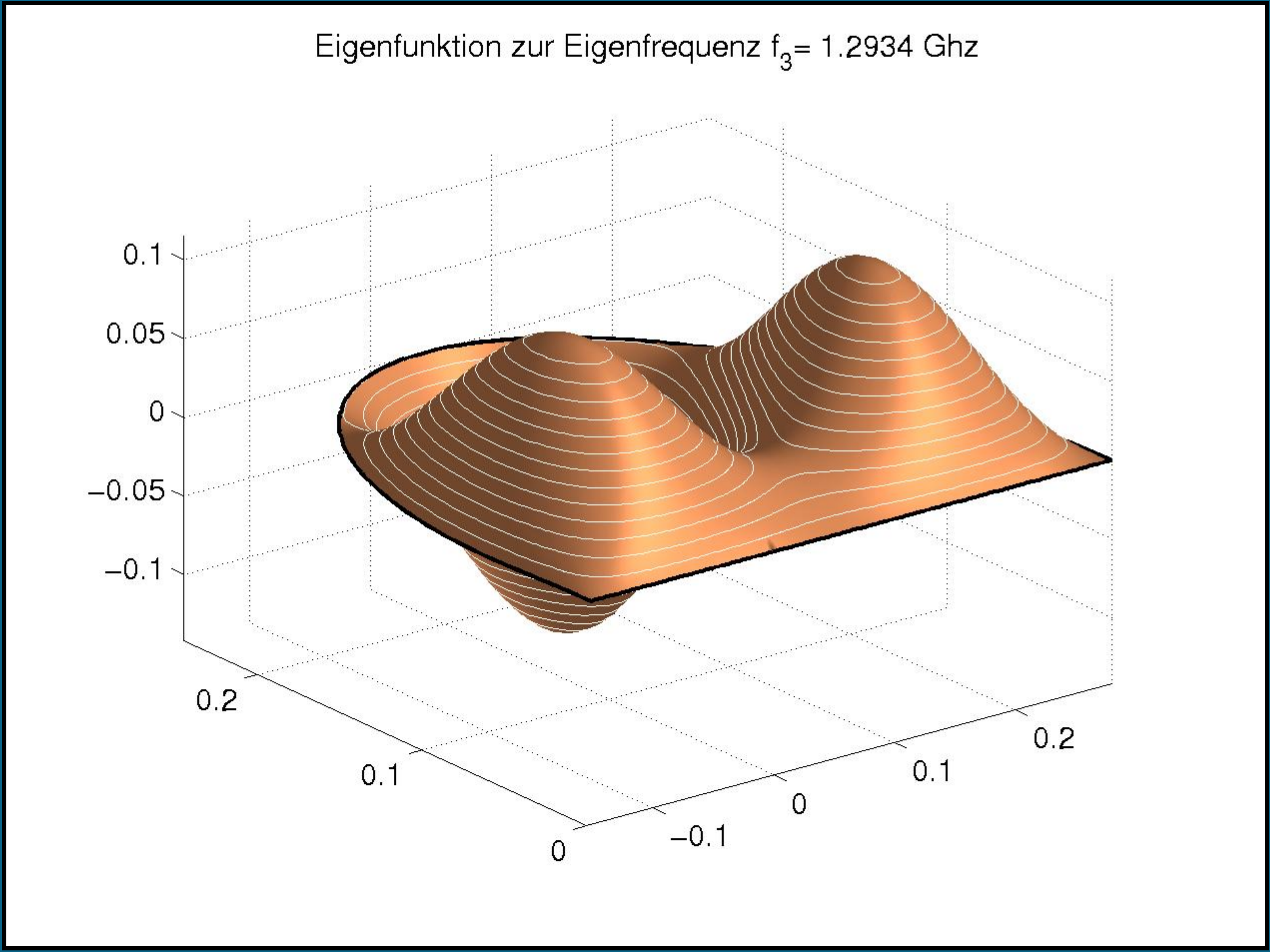
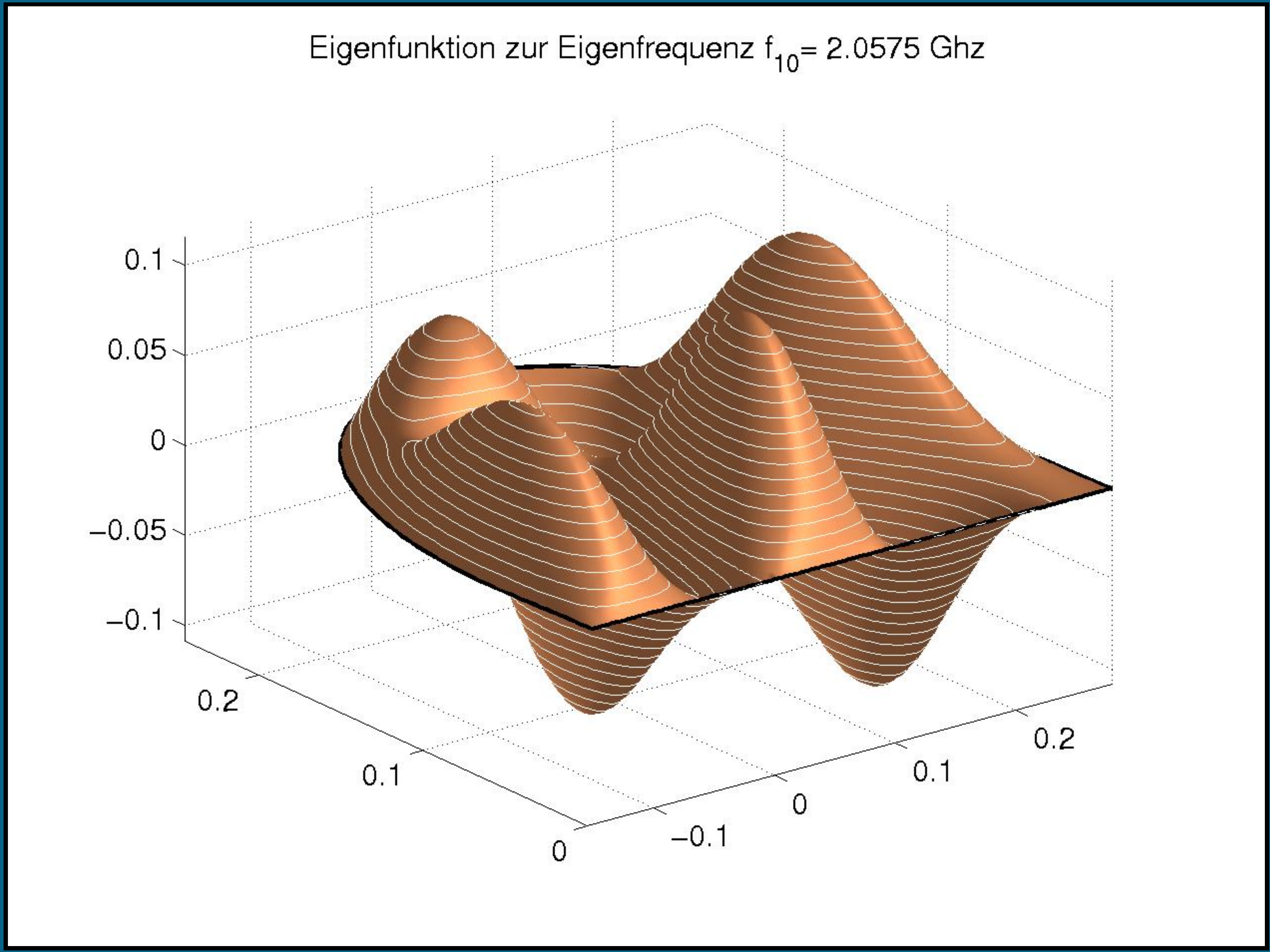
standard technique	200	100
multigrid	2000	100
web-splines	2000	1

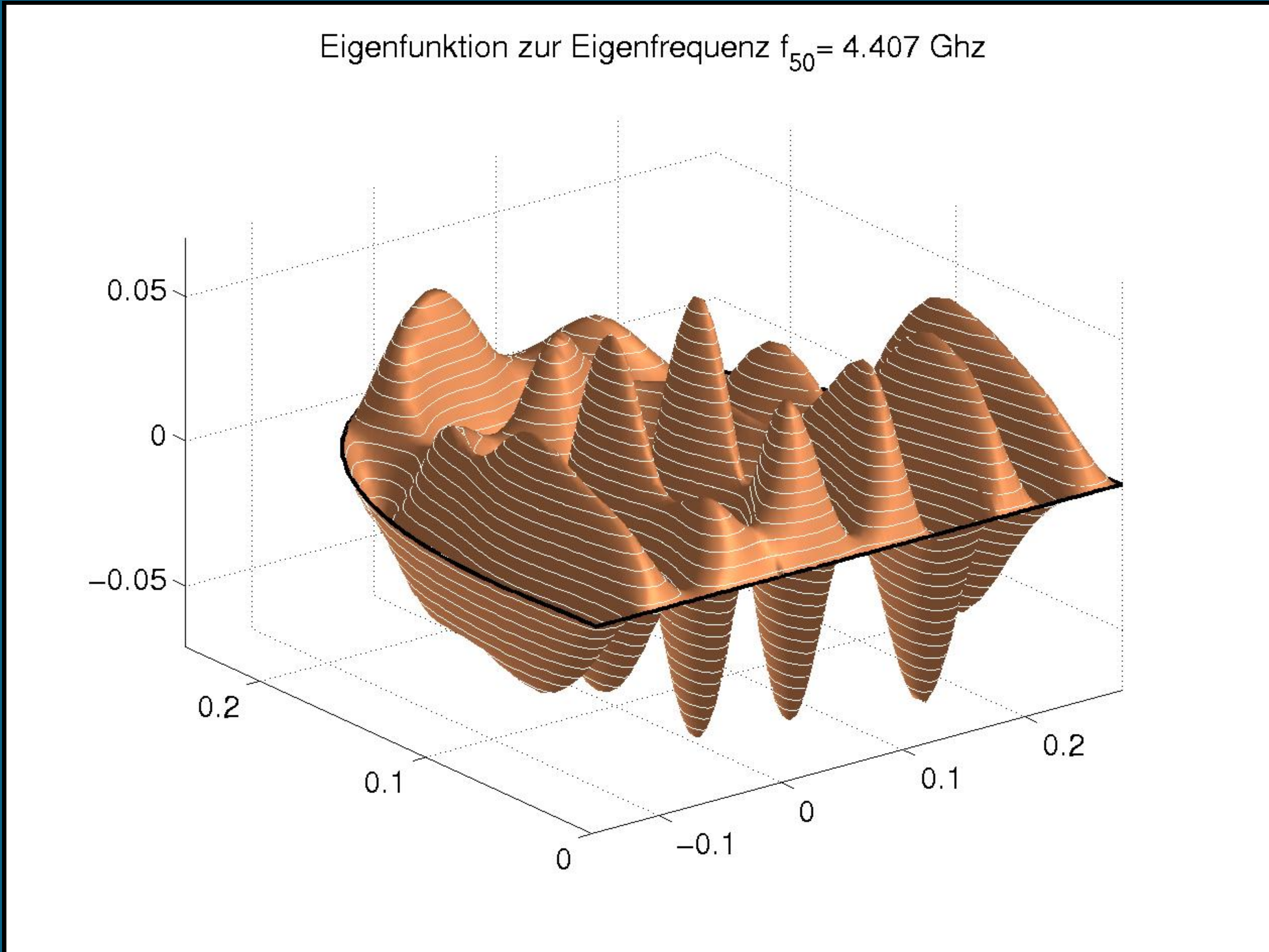


Eigenfunktion zur Eigenfrequenz f_1= 0.76072 Ghz
0.5
0.4
0.3
0.2
0.1
0
0.2
0.1
0
−0.1
0
0.1
0.2

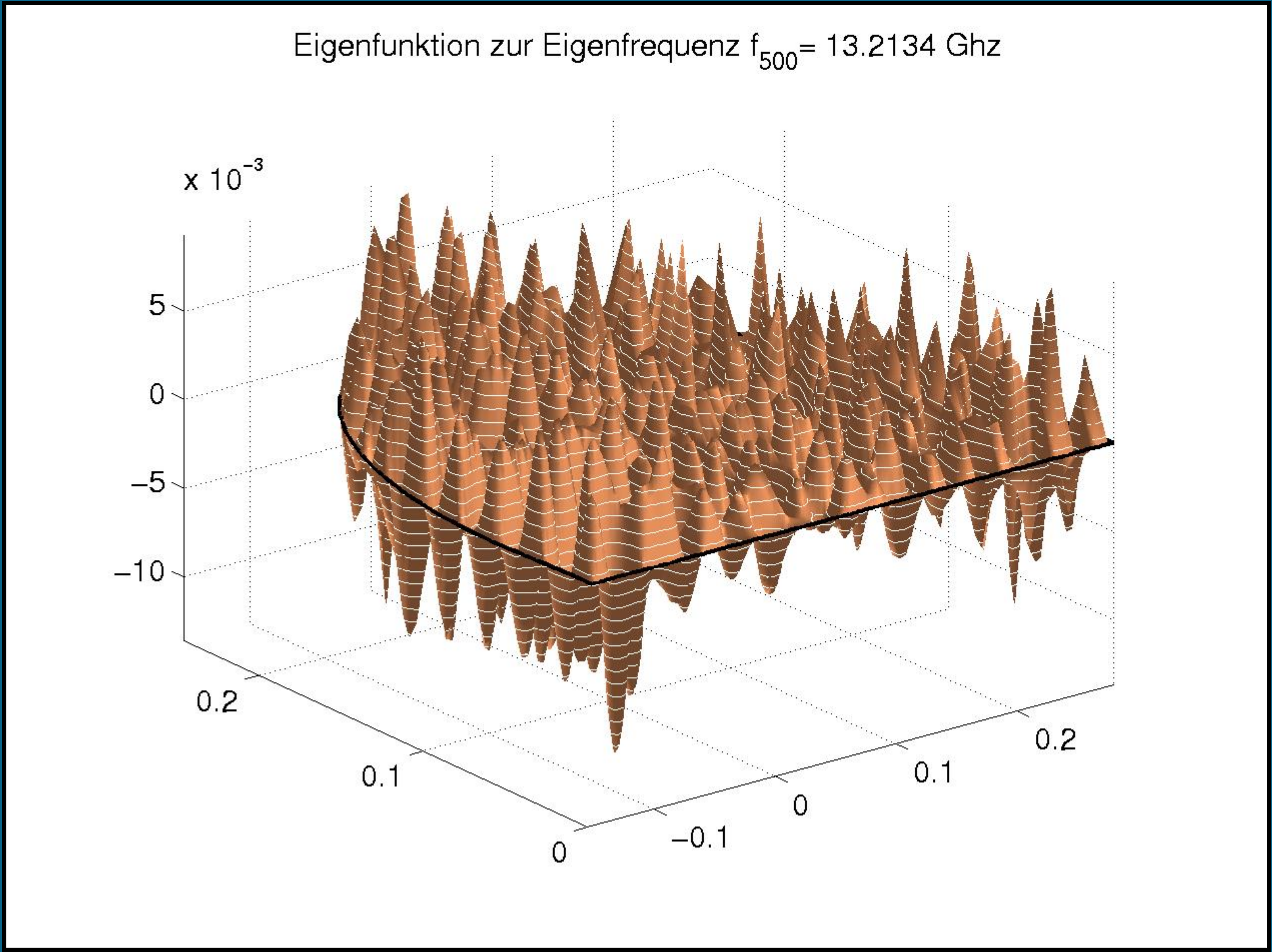

Eigenfunktion zur Eigenfrequenz f_2= 1.0419 Ghz
0.2
0.1
0
−0.1
0.2
0.1
0
−0.1
0
0.1
0.2


Eigenfunktion zur Eigenfrequenz f_3= 1.2934 Ghz
0.1
0.05
0
−0.05
−0.1
0.2
0.1
0
−0.1
0
0.1
0.2


Eigenfunktion zur Eigenfrequenz f10= 2.0575 Ghz
0.1
0.05
0
−0.05
−0.1
0.2
0.1
0
−0.1
0
0.1
0.2



Eigenfunktion zur Eigenfrequenz f50= 4.407 Ghz
0.05
0
−0.05
0.2
0.1
0
−0.1
0
0.1
0.2

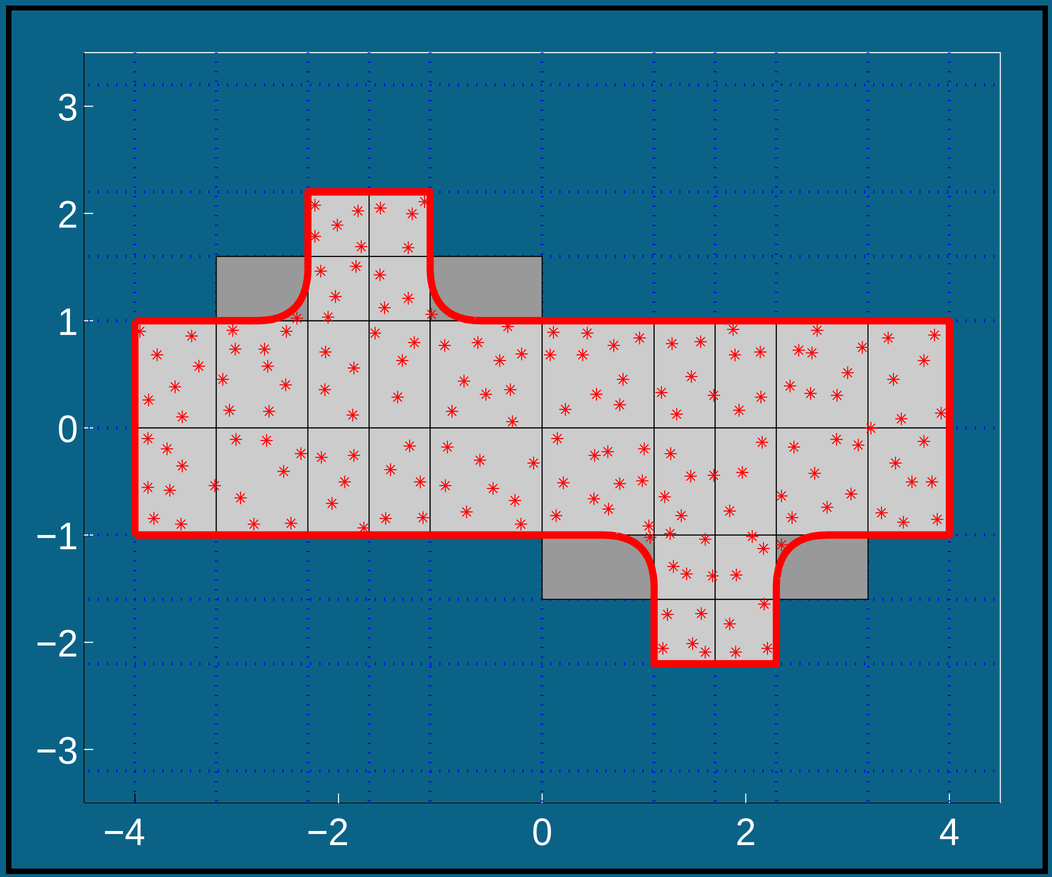


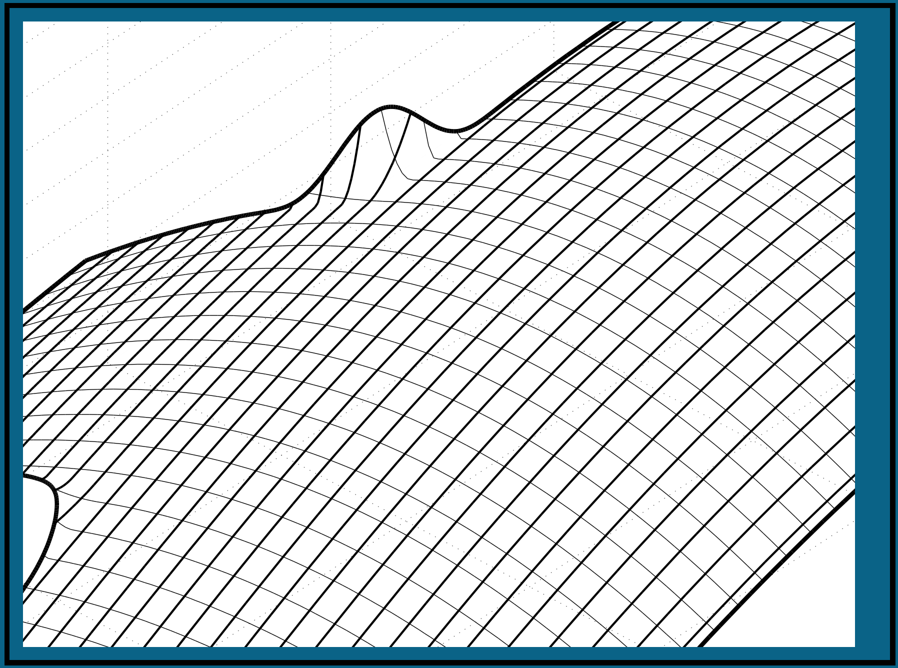
Eigenfunktion zur Eigenfrequenz f500= 13.2134 Ghz
x 10^-3
5
0
−5
−10
0.2
0.1
0
−0.1
0
0.1
0.2

Example 2: Reverse engineering

Given: Data points (x_i, y_i, z_i) *sampled on domain* Ω *from function* f.

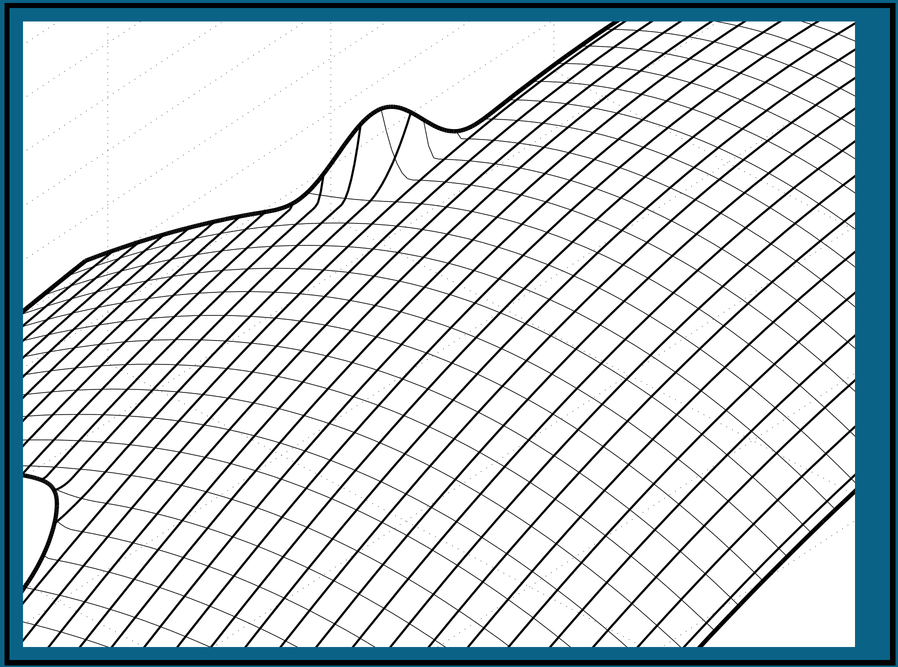
Sought: Function $g : \Omega \to \mathbb{R}$ *with* $g(x_i, y_i) \approx z_i$.

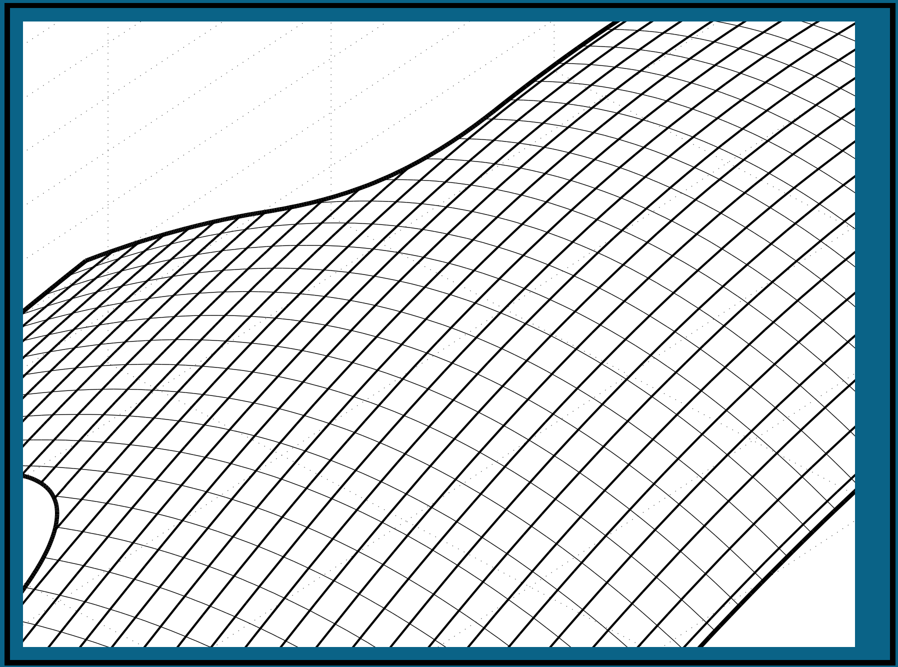




standard B-splines

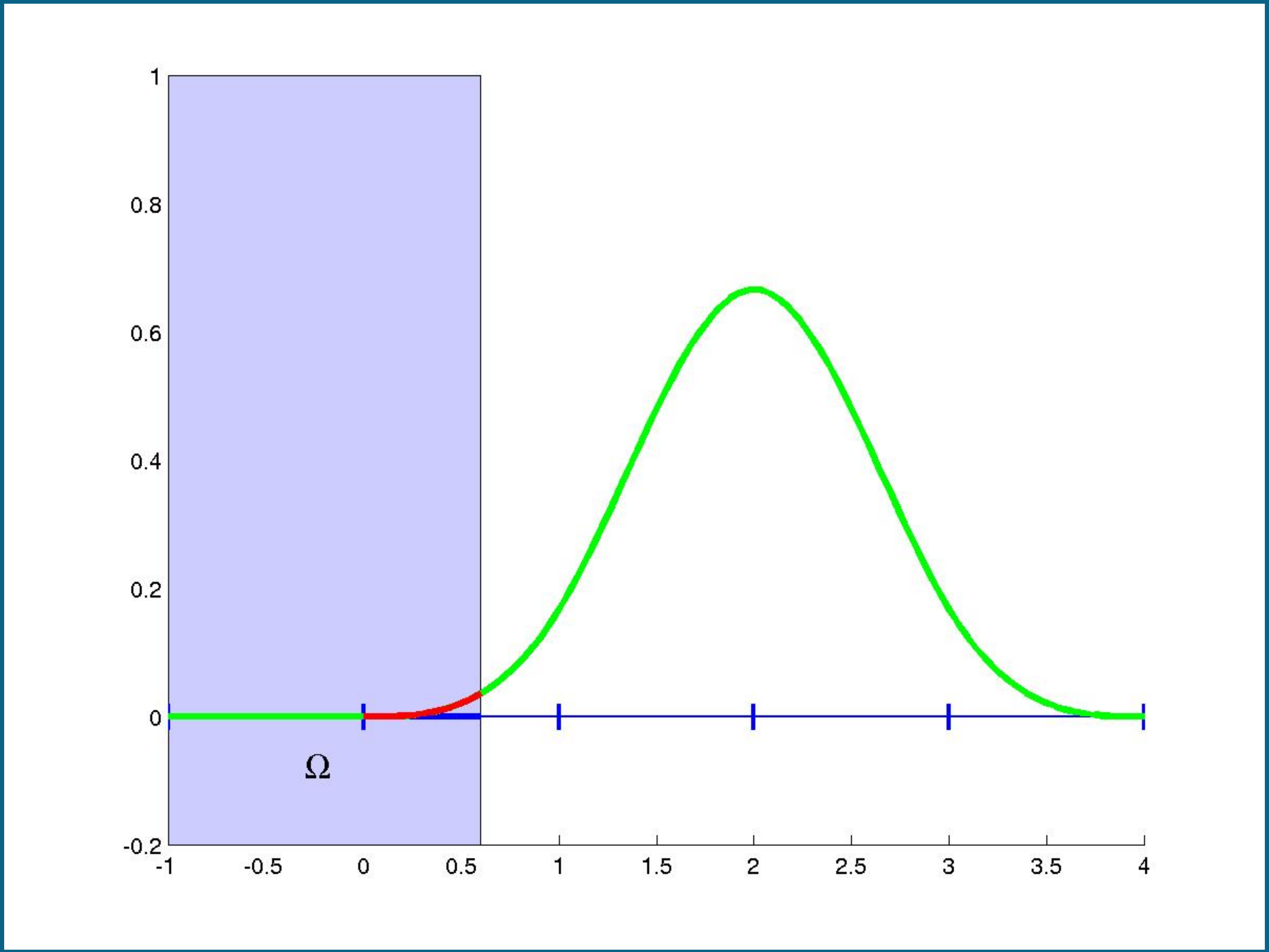
ℓ_∞-*error:*	$2.1e\text{-}4$
L_∞-*error:*	$2.8e\text{-}1$
condition number	$6.2e{+}13$





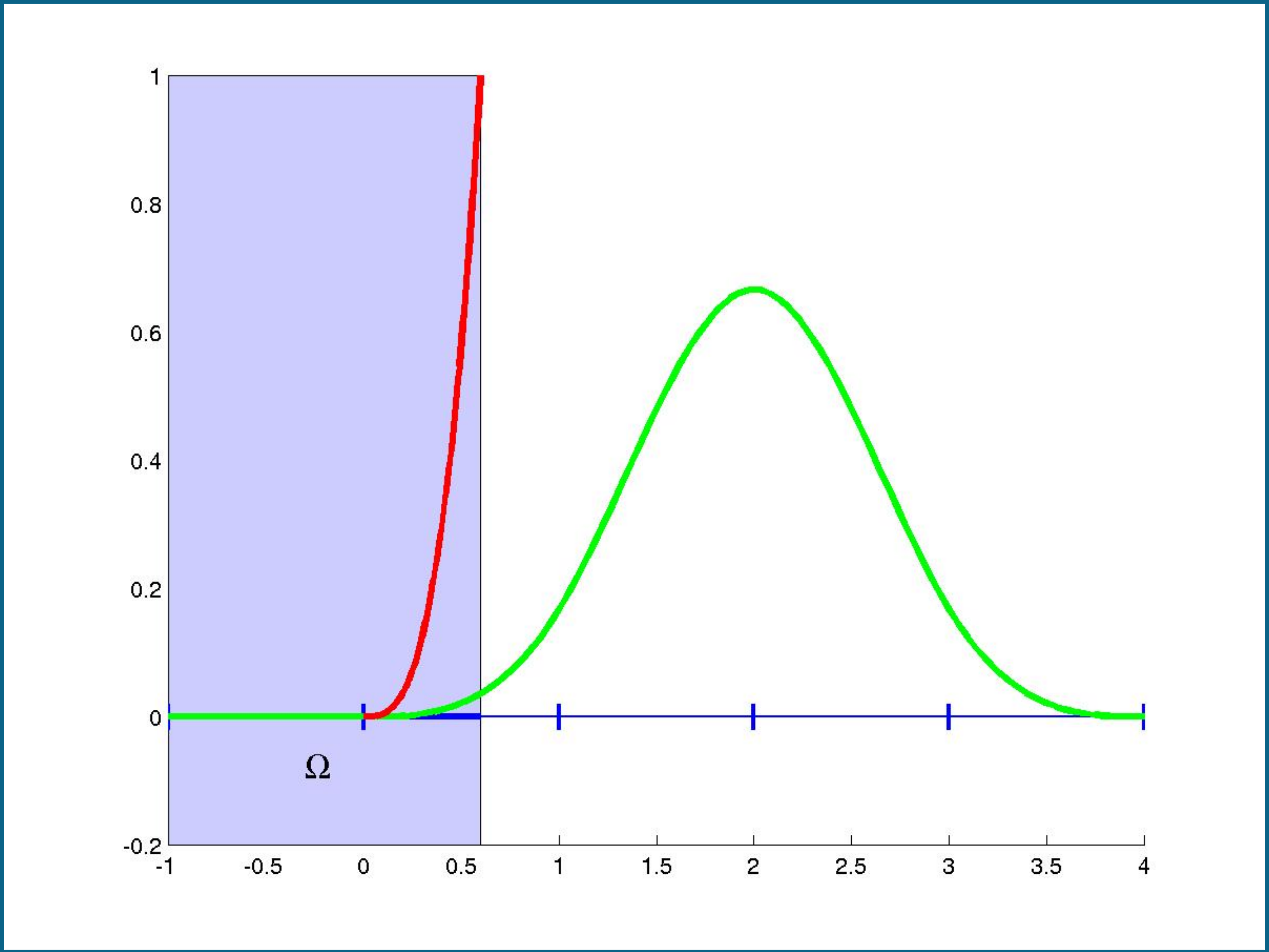
standard B-splines		*extended B-splines*	
ℓ_∞-*error:*	$2.1e\text{-}4$	ℓ_∞-*error:*	$2.2e\text{-}4$
L_∞-*error:*	$2.8e\text{-}1$	L_∞-*error:*	$2.3e\text{-}4$
condition number	$6.2e{+}13$	*condition number*	$7.7e{+}3$

Stabilization by normalization:



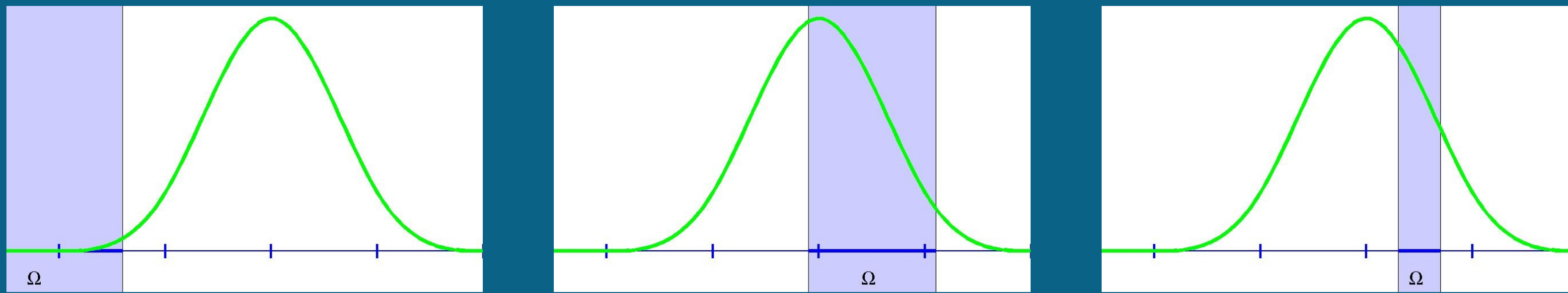
$$f(x) = b_i(x)$$

Stabilization by normalization:



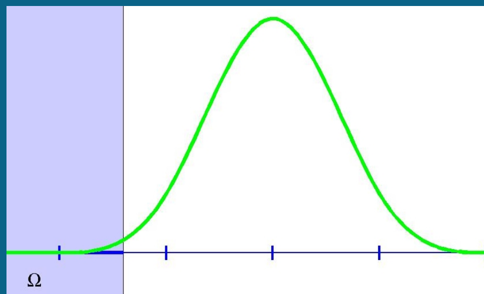
$$B_i(x) := \frac{b_i(x)}{\|b_i\|_\infty}$$

Stabilization in 1d

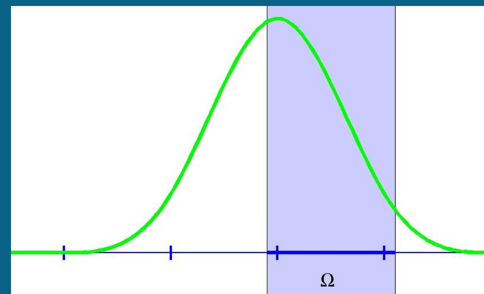


stable *stable* *not stable*

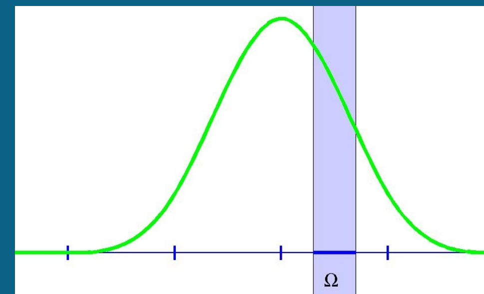
Stabilization in 1d



stable

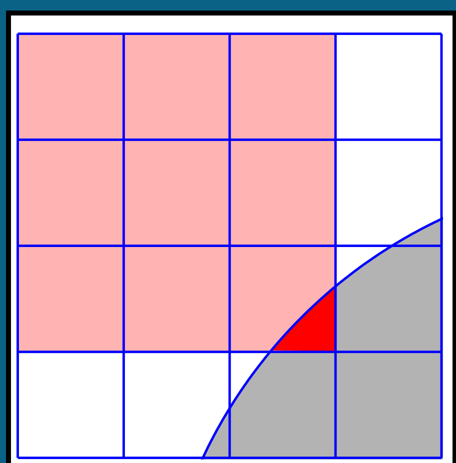


stable

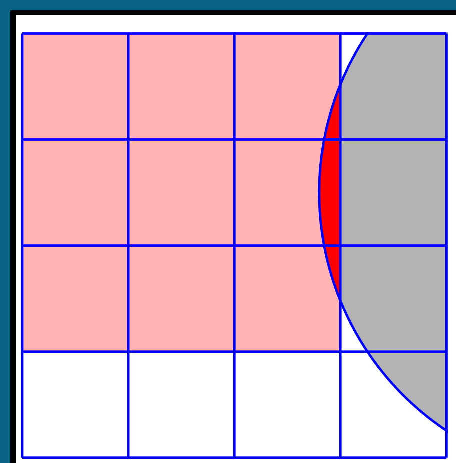


not stable

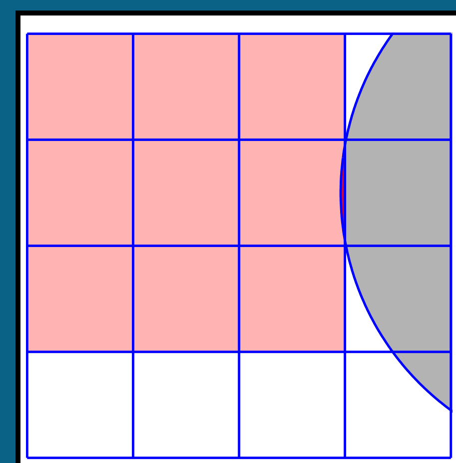
Stabilization in 2d



stable



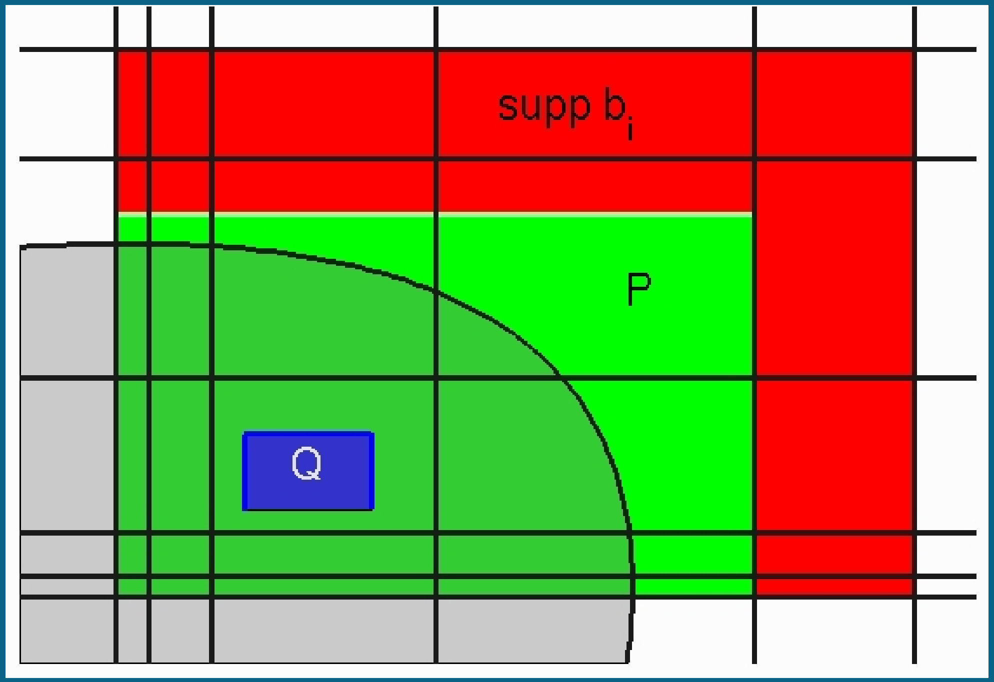
stable



not stable (rare)

Definition: *A normalized B-Spline B_i is called p-stable on $\Omega \subset \mathbb{R}^d$, if there exist intervals $Q \subset P \subset \operatorname{supp} b_i$ such that*

- *$Q \subset \Omega$ is contained in a grid cell*
- *P and $\operatorname{supp} b_i$ have a common corner*
- $n[Q] = [P]$
- $\|b_i\|_{p,\Omega} \leq \|b_i\|_{p,P}$



The index set of p-stable B-splines is denoted I_p.

Theorem The basis $\{B_i : i \in I_p\}$ is stable with respect to the p-norm. The constants depend only on n and p. Further, this basis provides full approximation power wrt. the L^2-norm.

Current work: Stokes equation

$$\begin{aligned} -\Delta u + \nabla p &= f \quad \textit{in } \Omega \\ \operatorname{div} u &= 0 \quad \textit{in } \Omega \\ u &= 0 \quad \textit{on } \partial\Omega \end{aligned}$$

Conjecture: *The Babuška-Brezzi-condition*

$$\inf_{\substack{p \in P \\ \|p\|_0 = 1}} \sup_{\substack{u \in U \\ \|u\|_1 = 1}} \int_\Omega p \operatorname{div} u \geq \beta > 0$$

is satisfied for

$$\deg P = n \times n, \quad \deg U = \begin{bmatrix} (n+1) \times n \\ n \times (n+1) \end{bmatrix}.$$

Conclusion:

- *web-splines*
  - *form a stable basis*
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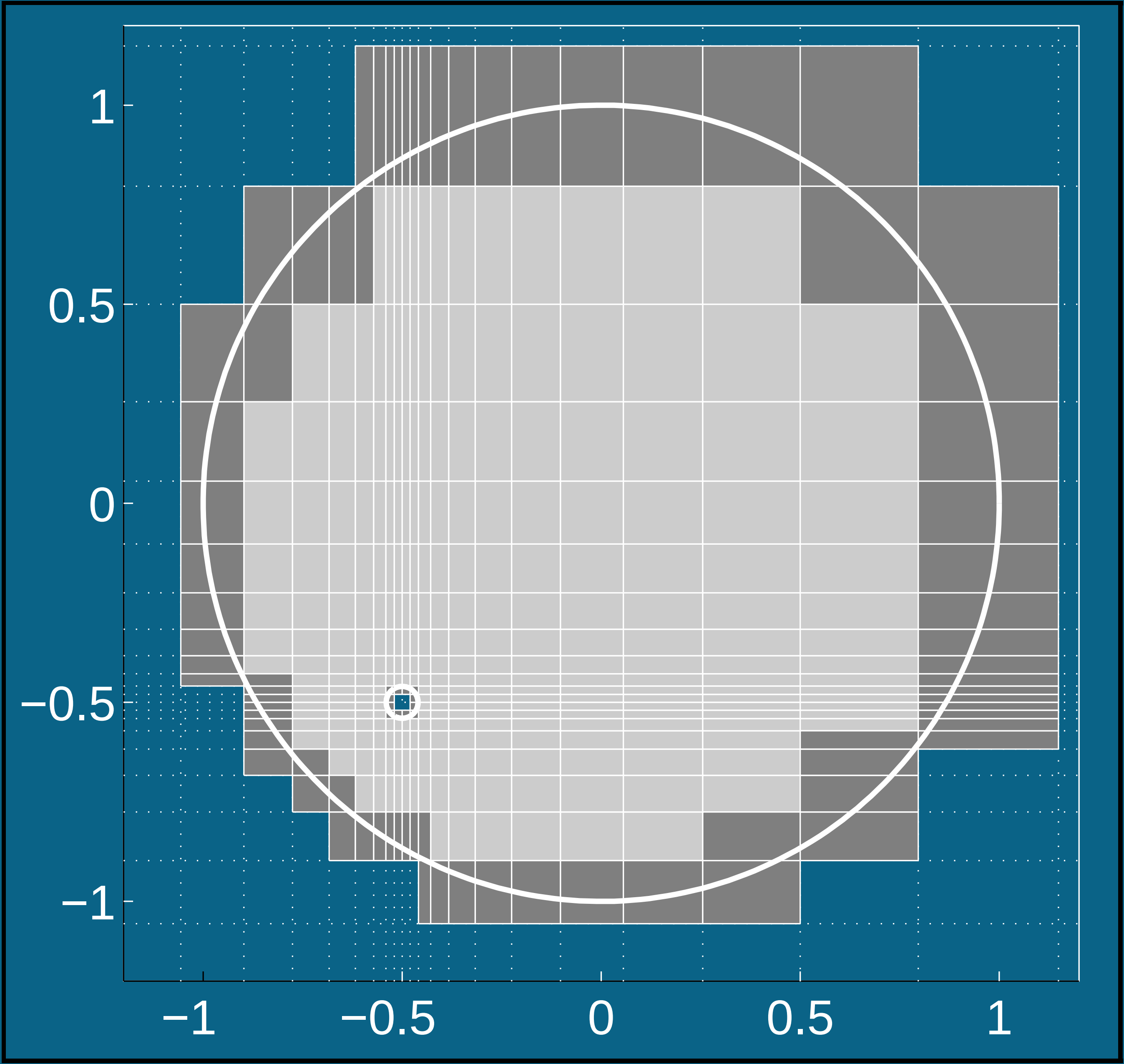
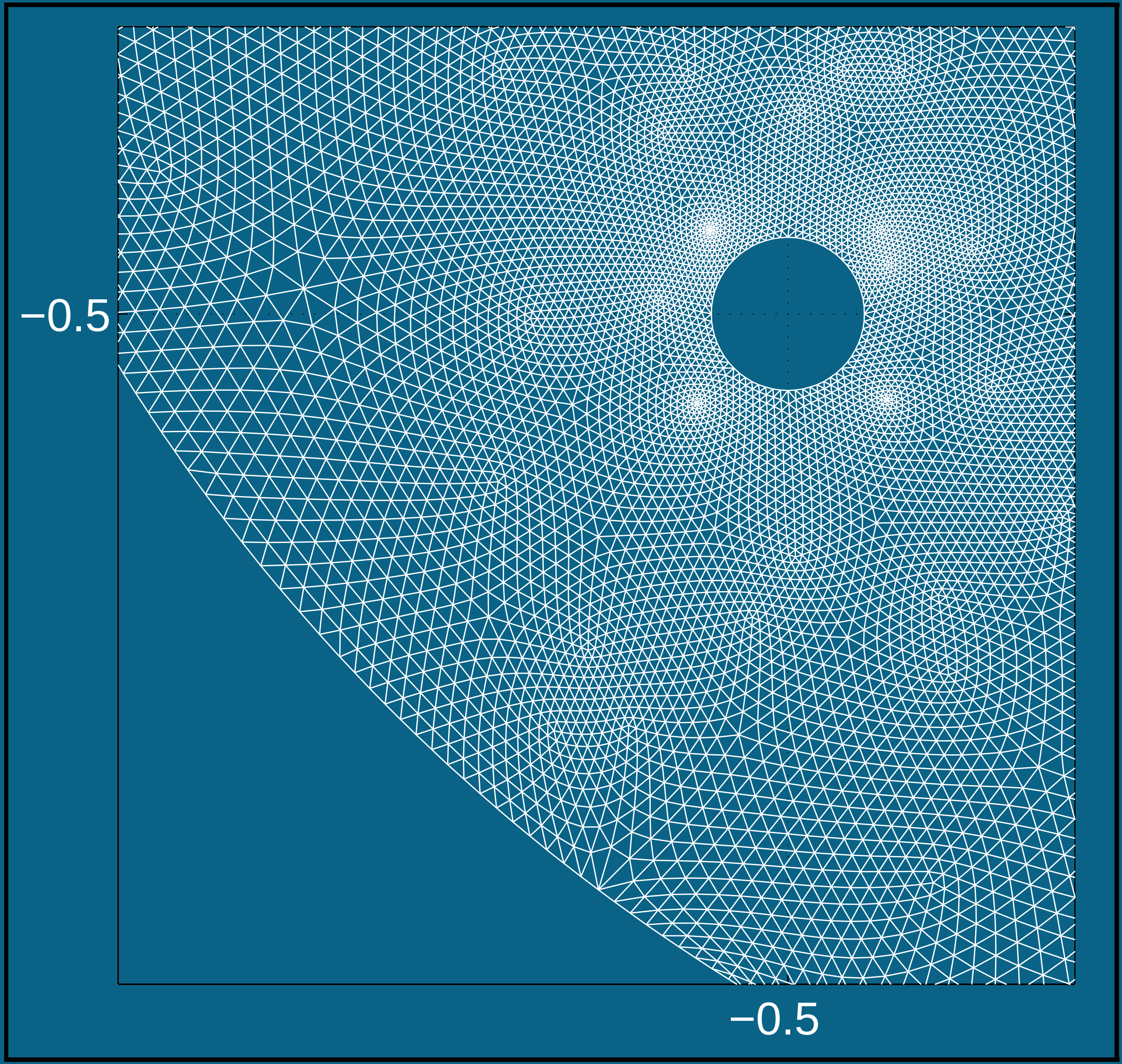
For further information, visit

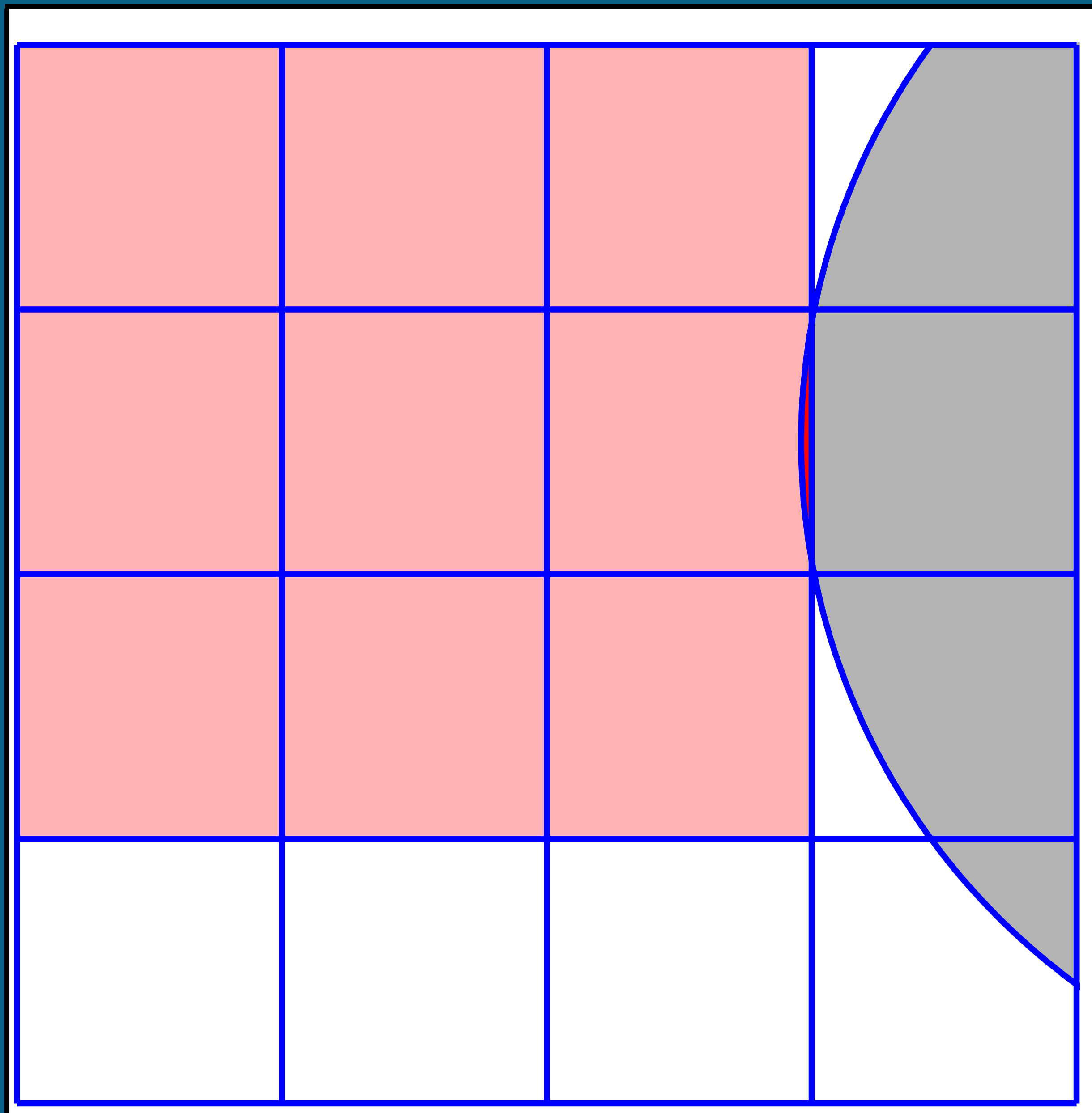
www.web-spline.de

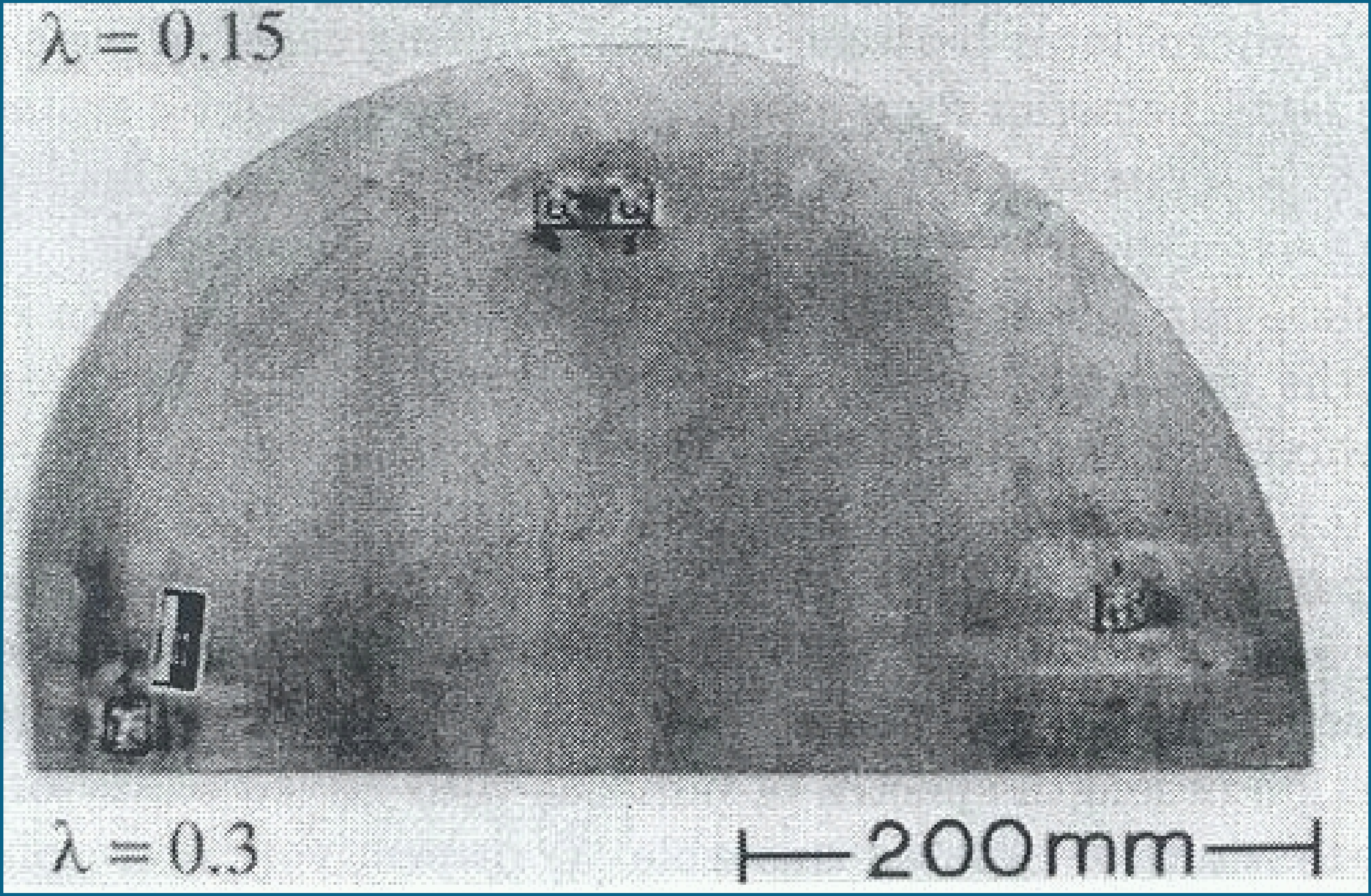


1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1



−0.5
−0.5




λ = 0.15
λ = 0.3
200mm