Finite Element Methods with Linear B-Splines

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FEM / B-Spline Web-Site (with U. Reif and J. Wipper): **http://www.web-spline.de/**

Avignon, July 5, 2006

Principal Features

- uniform grid
- one basis function per grid point
- **explicit vectorizing expressions for boundary integrals**
- fast discretization / linear precision
- small constant band width
- parallel algorithms
- multigrid solvers
- adaptive refinement

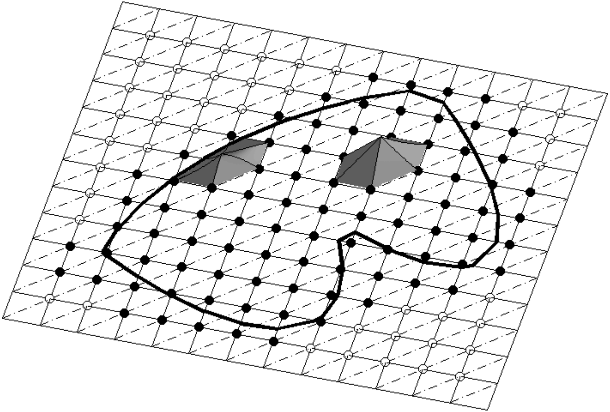
applications:

moderately accurate models

nonsmooth problems

time-sensitive simulations

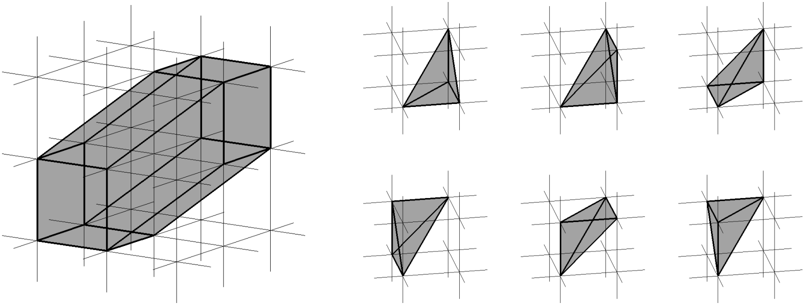
Finite Element Basis



domain $\quad D \approx D^h \; : \quad w^h(x) > 0$

weighted B-splines $\quad w^h b_k^h , \quad k_\nu \in \mathbb{Z}$

Linear Box-Splines [de Boor, Höllig, Riemenschneider, Springer 1993]

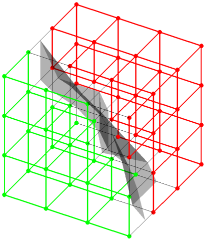


$$\Xi = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad B_{\Xi}(x) = \int_0^1 \chi_{[0,1]^3}\,(x - t(1,1,1))\,dt$$

Lagrange basis $\quad b_k^h(x) = B_{\Xi}\,(x/h - k)\,, \quad k \in \mathbb{Z}^3$

Data Structure

values on grid of bounding box



$$w^h(0:k_1, 0:k_2, 0:k_3)$$

$$p^h(0:k_1, 0:k_2, 0:k_3)$$

finite element approximation (Dirichlet boundary conditions)

$$u^h = w^h p^h = \left(\sum_k w_k^h b_k^h\right)\left(\sum_k p_k^h b_k^h\right)$$

evaluation on grid tetrahedron S

$$w_S^h \times p_S^h \in \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \text{product of linear functions}$$

Stability [cf. results by Reif and Mößner] and Accuracy

Condition of the basis:

$$\left\| u^h \right\|_{0,D^h} \asymp \left\| \Gamma P^h \right\|_0$$

with Γ the diagonal matrix of normalization constants $\gamma_k = \left\| w^h b_k^h \right\|_{0,D^h}$.

Approximation order:

$$\left\| u - u^h \right\|_{1,D \cap D^h} \preceq h \left\| u \right\|_{2,D}$$

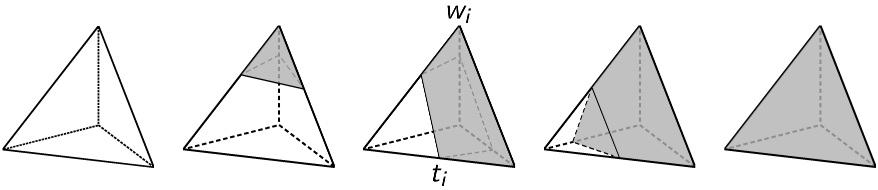
for a domain D with smooth boundary and $u \in H^2(D) \cap H_0^1(D)$.

Numerical Integration

Ritz-Galerkin integrals

$$\int_{S \cap D^h} a\left(w^h b_k^h, w^h b_{k'}^h\right) = I\left(w_S^h, k, k'\right)$$

intersection patterns



Poisson bilinear form

$$I \ : \text{rational function of } w_S^h \text{ or } t_S^h$$

(automatically generated, $6 * 16 * 10$ cases)

Expressions for Ritz-Galerkin Integrals

no intersection, full box, common factor $1/120$:

$$
\begin{aligned}
&24\, w_8^{\,2} - 8\, w_1\, w_3 + 48\, w_1^{\,2} + 8\, w_5^{\,2} + 24\, w_1\, w_8 + 8\, w_1\, w_6 \\
&-4\, w_5\, w_6 - 8\, w_1\, w_5 - 4\, w_8\, w_6 - 4\, w_3\, w_4 + 12\, w_6^{\,2} - 4\, w_5\, w_7 \\
&+8\, w_1\, w_7 + 8\, w_3^{\,2} - 4\, w_4\, w_8 + 12\, w_4^{\,2} - 4\, w_6\, w_2 + 8\, w_1\, w_4 \\
&+12\, w_7^{\,2} + 8\, w_2^{\,2} - 8\, w_2\, w_1 - 4\, w_3\, w_7 - 4\, w_2\, w_4 - 4\, w_7\, w_8
\end{aligned}
$$

intersection, one tetrahedron, common factor $w_1^2/120$:

$$
\begin{aligned}
&-\frac{t_2^{\,2} t_1^{\,2}}{t_3} + t_1^{\,2} t_2 - 2\, t_1\, t_2\, t_3 - 4\, \frac{t_1^{\,3} t_3}{t_2} - 2\, \frac{t_1^{\,2} t_3^{\,2}}{t_2} \\
&-5\, t_1\, t_3 + 3\, t_1^{\,2} t_3 + 4\, t_1^{\,3} - 10\, t_1^{\,2} + t_1\, t_3^{\,2} - 2\, \frac{t_1^{\,3} t_2}{t_3} \\
&+5\, \frac{t_1^{\,2} t_2}{t_3} + 10\, \frac{t_1^{\,2} t_3}{t_2}
\end{aligned}
$$

K-Form [J. Koch, Dissertation 1995]

sparse polynomial expressions

$$2xy + 3x^2y^2z + \dots$$
$$4x^2y^2z + 5xyz + \dots$$

simplifying quadratic substitutions

$$u = xy\,, \quad v = uz\,, \quad w = uv$$

$\rightarrow$ linear expressions

$$2u + 3w + \dots$$
$$4w + 5v + \dots$$

Example of K-Form Substitutions

$$
-\frac{t_2^2 t_1^2}{t_3} + t_1^2 t_2 - 2\, t_1 t_2 t_3 - 4\,\frac{t_1^3 t_3}{t_2} - 2\,\frac{t_1^2 t_3^2}{t_2}
$$
$$
-5\, t_1 t_3 + 3 t_1^2 t_3 + 4 t_1^3 - 10\, t_1^2 + t_1 t_3^2 - 2\,\frac{t_1^3 t_2}{t_3}
$$
$$
+5\,\frac{t_1^2 t_2}{t_3} + 10\,\frac{t_1^2 t_3}{t_2}
$$

15 substitutions

$$
-t_{14} + t_{12} - 2t_{13} - 4t_{17} - 2t_{18} - 5t_7 + 3t_{15}
$$
$$
+4t_{10} - 10t_6 + t_{16} - 2t_{11} + 5t_8 + 10t_9
$$

substitutions could be used simultanously for all 36 cases of (k, k')

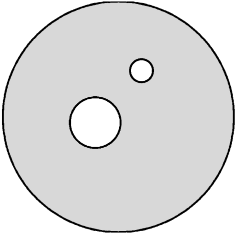
$\rightarrow$ 27 Substitutions for 315 products for the non-intersection case

Performance

Eigenvalue Problem

$$-\Delta u = \lambda u \text{ in } D\,, \quad u = 0 \text{ on } \partial D$$

Lanczos method with dynamic multigrid



unknowns:	6561
CPU time (sec.):	581
number of computed eigenvalues:	1000
$\frac{\text{time}}{\text{eigenvalue}}$ (sec.):	0.581

Computed on Pentium 4 PC, 3.20GHz, 2GB memory

Program Demo

$$-\Delta u = f \text{ in } D\,, \quad u = 0 \text{ on } \partial D$$

unit disc with random holes
dynamic multigrid solver
FORTRAN 90 code
MATLAB interface for visualization

holes	$f = 4$		$f = e^x((1+x)\cos y - y \sin y)$	
0	6 grids	8 grids	6 grids	8 grids
2	6 grids	8 grids	6 grids	8 grids
4	6 grids	8 grids	6 grids	8 grids

Next Steps [in cooperation with U. Küster, HLRS]

- porting programs to NEC SX-8 Cluster (72 Nodes with 8 CPUs each)
- large scale tests of the vectorization
- parallel processing of different intersection types
- completion of the 3D implementation
- improvement of automatic algebraic simplifications

web-sites:

`http://www.web-spline.de/`
`http://www.mathematik-online.org/`

books:

Finite Element Methods with B-Splines, SIAM, 2003
Handbook of Computer Aided Geometric Design, Elsevier 2002