Finite Element Analysis with B-Splines: Weighted and Isogeometric Methods

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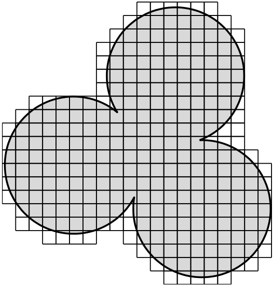
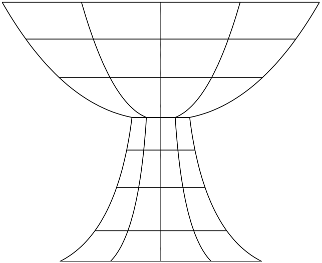


http://www.web-spline.de/



http://users.ices.utexas.edu/~hughes/

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Contributions: C. Apprich and M. Boßle

Finite Element Method

Variational problem

$$\mathcal{Q}(u) = \int_D F(x, u, \nabla u, \ldots)\, dx \to \min$$

$u \in H$, Hilbert space incorporating essential boundary conditions
Approximation

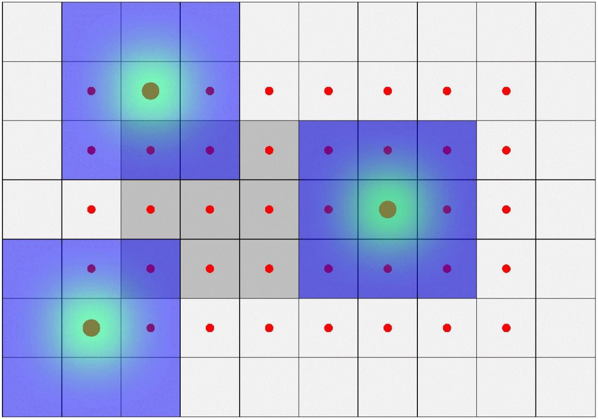
$$u_h = \sum u_k B_k \in \mathbb{B}_h \subset H \,:\; \mathcal{Q}(u_h) = \min_{v_h \in \mathbb{B}_h} \mathcal{Q}(v_h)$$

Poisson problem

$$\begin{aligned} F &= \frac{1}{2} |\nabla u|^2 - fu \\ H &: \quad u,\ \nabla u \in L_2(D)\,, \quad u_{|\partial D} = 0 \end{aligned}$$

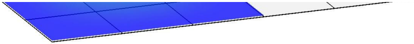
Uniform Splines

$$u_h(x) = \sum_{k \sim R} u_k b_k(x), \quad x \in R$$



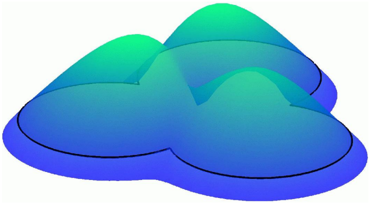
• nodes x^k, grid position k

Uniform Splines

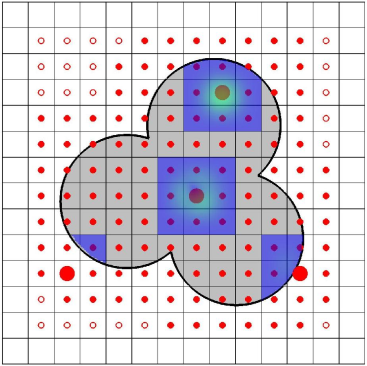


k=(1,1)

Weighted B-Splines



weight function: $D : w > 0$



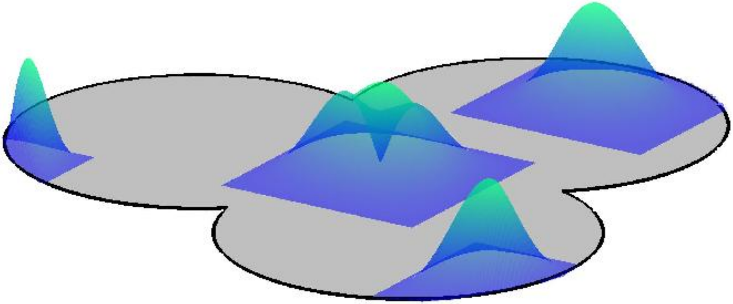
$B_k = w b_k$

Stabilization

extension coefficients: $e_{i,k} \in \mathbb{Z}$:

$$B_i = \sum_{k \sim i} e_{i,k} (w b_k)$$

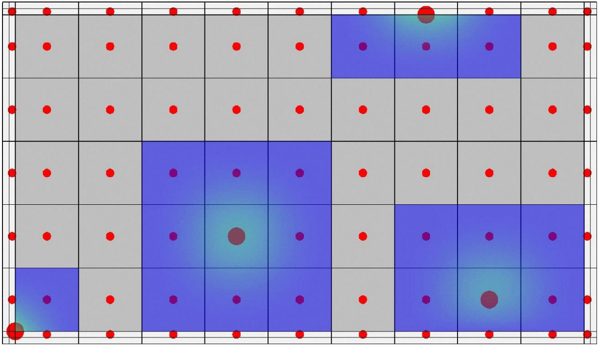
Weighted B-Splines



normalized: $B_k = \alpha_k w b_k$

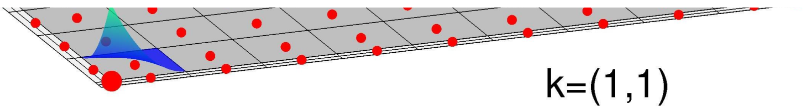
Boundary Conforming Splines

$$u_h(\xi) = \sum_{k \sim R} u_k b_k(\xi), \quad \xi \in R$$

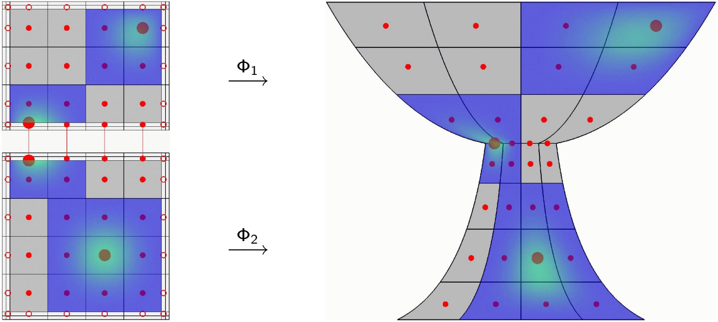


• nodes ξ^k, grid position k

Boundary Conforming Splines



Isogeometric B-Splines



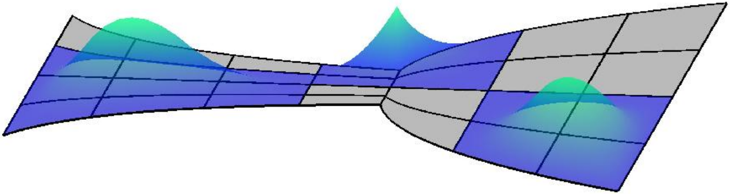
Spline parametrization

$$D = \bigcup \Phi_\nu(R_\nu)\,, \quad \Phi_\nu = \sum_k \Phi_{\nu,k} b_k$$

Basis functions

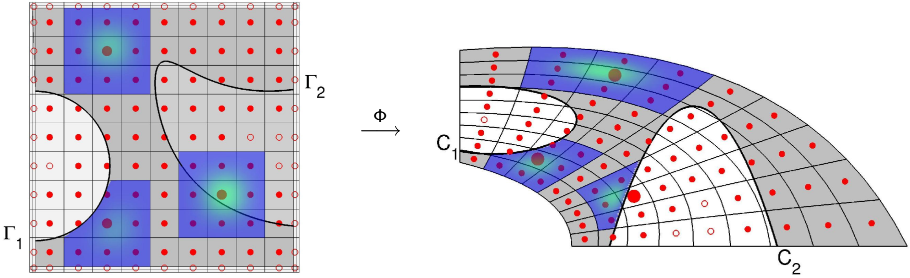
$$B_{k,\nu}(x) = b_{k,\nu}(\xi), \quad x = \Phi_\nu(\xi)$$

Isogeometric B-Splines



normalized: $B_{k,\nu}(x) = \alpha_k b_{k,\nu}(\xi)$

Weighted Isogeometric B-Splines



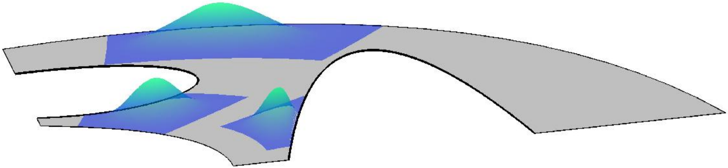
Trim curves

$$\Gamma_1 : w_1(\xi) = 0, \quad C_2 : w_2(x) = 0$$

Basis

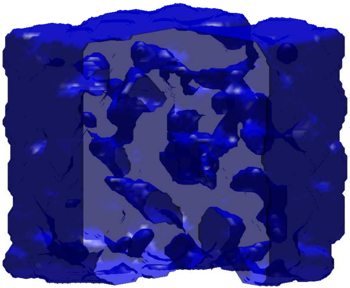
$$B_k(x) = w_1(\xi)\, b_k(\xi) w_2(x)\ , \quad x = \Phi(\xi)$$

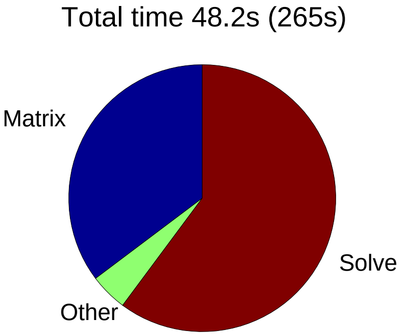
Weighted Isogeometric B-Splines



normalized: $B_k(x) = \alpha_k w_1(\xi)\, b_k(\xi) w_2(x)$

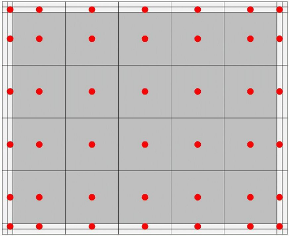
Weighted B-Splines for Random Domains



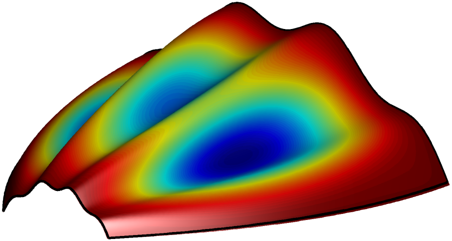


Problem type:	Dirichlet
Discretization:	192,100,033 (577^3) unknowns, 7 grids
Solver:	Dynamic-Multigrid-Jacobi (10 smoothings, $\omega = 0.88$)
Residual:	5.462E-09 ($<$ 1E-8)
Iterations:	12.1289
Machine:	NEC-SX8, 8 CPUs (1 Node), 103 GB Memory

Isogeometric B-Splines for Shells



$\overset{\Phi}{\longrightarrow}$



Approximation of the displacement (Koiter's Model)

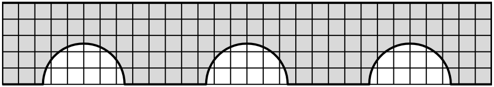
$$\vec{u}_h(x) = \sum_k \vec{u}_k b_k(\xi), \qquad \xi = \Phi^{-1}(x)$$

Bilinear form

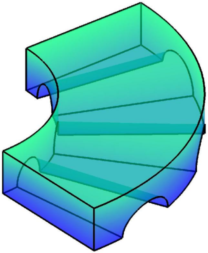
$$a(\vec{u}_h, \vec{v}_h) = \int_D E^{ijkl} \left(\gamma_{ij}(\vec{u}_h)\gamma_{kl}(\vec{v}_h) + \varrho_{ij}(\vec{u}_h)\varrho_{kl}(\vec{v}_h)\right)$$

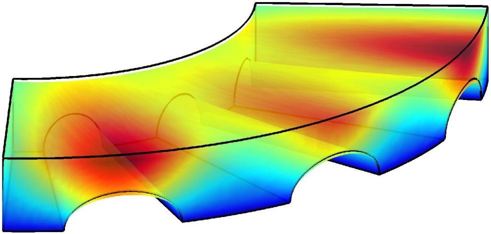
coefficients E^{ijkl} depending on the geometry of the shell,
strain tensor γ_{ij}, change of curvature tensor ϱ_{ij} (normalized)

Weighted Isogeometric B-Splines for Linear Elasticity



$\xrightarrow{\Phi}$





color: norm of of displacement

concrete
density:
$\varrho = 2.4\,\mathrm{t/m^3}$
Young's modulus:
$E = 50\,\mathrm{kN/mm^2}$
Poisson's ratio: 0.2

Pros and Cons

- efficiency of b-spline algorithms
- compatibility with CAD/CAM applications
- arbitrary order and smoothness
- high accuracy with few parameters
- exact fulfilment of boundary conditions
- simple data structure
- multigrid solvers and adaptive refinement
- vectorization and parallelization

- integration over irregularly shaped boundary cells (W)
- partition of domain into deformed rectangles/cubes (I)
- construction of weight functions for parametric domains (W)
- incorporation of trim curves/surfaces (I)