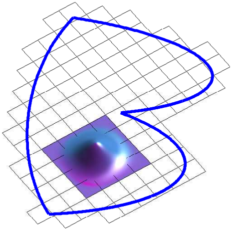
Finite Element Methods with WEB-Splines

Klaus Höllig



http://www.web-spline.de

Cooperation partners: Prof. Dr. U. Reif, Dr. J. Wipper

Contributions by:
Dr. A. Fuchs, J. Hörner, Dr. M. Pfeil

Finite Element Method

boundary value problem

$$-\Delta u = f \text{ in } D, \qquad u = 0 \text{ on } \partial D$$

weak form

$$Q(u) = \frac{1}{2}\int_D |\text{grad}\, u|^2 - \int_D fu \to \min, \qquad u \in H$$

Ritz-Galerkin approximation

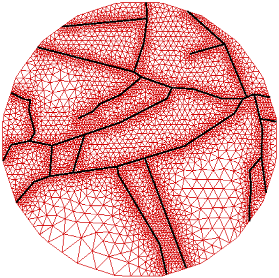
$$Q(u_h) \to \min, \qquad u_h \in V_h$$

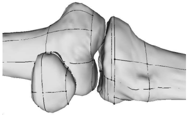
Céa Lemma

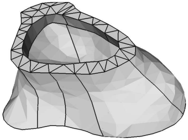
$$\|u - u_h\| \lesssim \inf_{v_h \in V_h} \|u - v_h\|$$

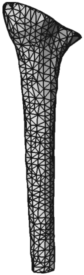
Standard Finite Elements

ART Triangulations (Dr. A. Fuchs)



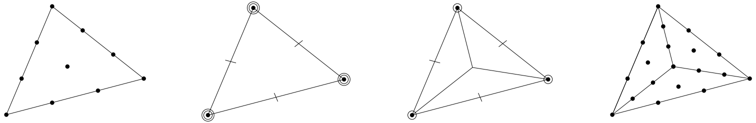






Element types for triangulations

Lagrange Argyris Clough-Tocher Lagrange



$(3, 0, 10)$ $(5, 1, 21)$ $(3, 1, 12)$ $(3, 0, 20)$

(degree, smoothness, dimension)

Splines: Applications and Techniques

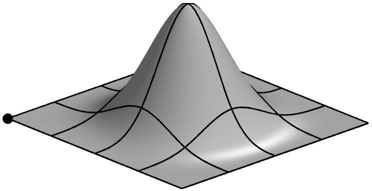
- Approximation and Data Fitting
- Automated Design and Manufacturing
- Numerical Analysis
- N-Width and Optimal Recovery
- Computer Graphics and Image Processing
- Computer Aided Geometric Design
- Finite Elements

Λ splines, β-splines, ν-splines, ω-splines, τ-splines, A-splines, ARMA splines, B-splines, Bernoulli splines, BM-splines, box-splines, cardinal splines, Catmull-Rom splines, D^m-splines, Dirichlet splines, discrete splines, E-splines, elliptic splines, exponential Euler splines, exponential box splines, fundamental splines, g-splines, Gibbs-Wilbraham splines, $H^{m,p}$-splines, harmonic splines, Helix splines, Hermite splines, Hermite-Birkhoff splines, histosplines, hyperbolic splines, Inf-convolution splines, K-splines, L-monosplines, L-splines, Lagrange splines, LB-splines, Legendre splines, $Lg-$splines, M-splines, metaharmonic splines, minimal-energy splines, monosplines, natural splines, NBV-splines, NURBS, ODR splines, PDL_g splines, perfect splines, PL_{-g} splines, $pLg-$splines, polyharmonic splines, Powell-Sabin splines, pseudo splines, Q-splines, Schoenberg splines, simplex splines, smoothing splines, super splines, thin-plate splines, triangular splines, trigonometric splines, Tschebyscheff splines, TURBS, v-splines, variation diminishing splines, vertex splines, VP-splines, web-splines, Wilson-Fowler splines, X-splines, ...

Multivariate Splines

Multivariate B-splines

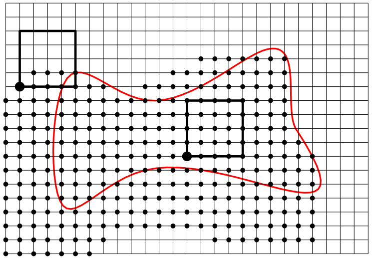
$$b_k(x_1, \ldots, x_m) = \prod_{\nu=1}^{m} b_{k_\nu}(x_\nu)$$



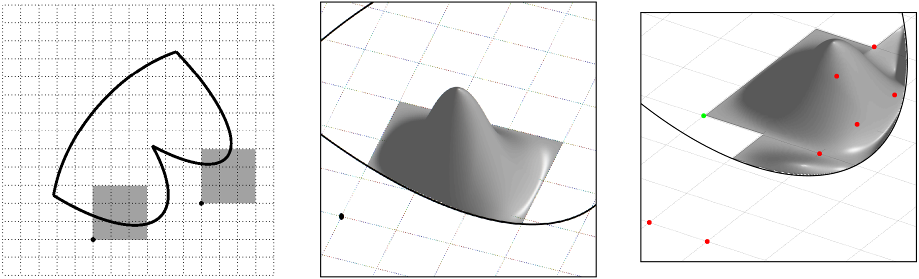
Linear combinations of relevant B-splines

$$v_h(x) = \sum_{k \in K} c_k b_k$$

$$K : k \sim D$$



B-Splines as Finite Elements



homogeneous boundary condition via weight function

$$b_k \longrightarrow \quad w\, b_k$$

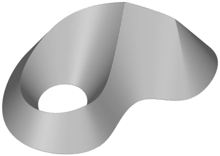
stability via extension

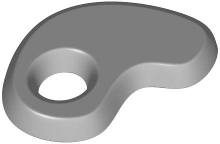
$$b_i \longrightarrow \quad b_i + \sum_{j \in J(i)} e_{i,j} b_j$$

Weight Function

$$w(x) \asymp \operatorname{dist}(x, \partial D), \qquad \text{smooth on } D$$

- Smoothed distance function: $w(x) = 1 - \max\left(0, 1 - d(x, \partial D)/\delta\right)^{\ell}$





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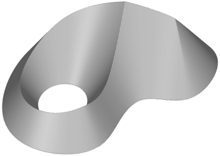

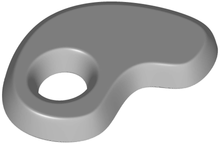

- Implicit representation of ∂D:

$$w(x) = 1 - x_1^2 - x_2^2 \quad \text{for} \quad D: \ |x| < 1$$

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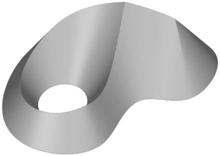

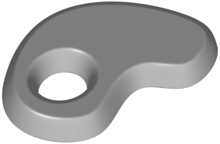

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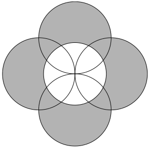
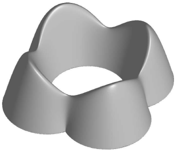
- R-functions: $r_c(w) = -w$, $r_{\cup/\cap}(w_1, w_2) = w_1 + w_2 \pm \sqrt{w_1^2 + w_2^2}$





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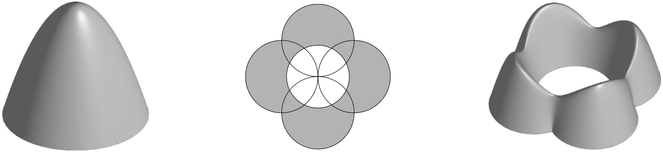
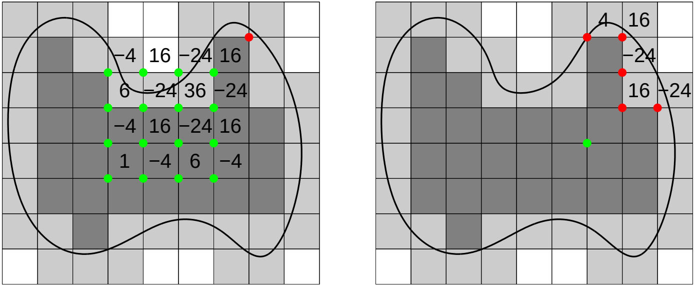


- Spline approximation

Extension

Coupling outer with inner B-splines

$$B_i = b_i + \sum_{j \in J(i)} e_{i,j} b_j, \quad e_{i,j} = \prod_{\nu=1}^{m} \prod_{\substack{\mu=\ell_\nu \\ \mu \neq i_\nu}}^{\ell_\nu+n-1} \frac{j_\nu - \mu}{i_\nu - \mu}$$

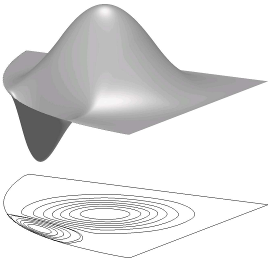


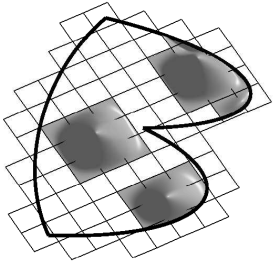
$I(j)$: nearest $(n+1)^m$-array of inner indices

$J(i)$: dual sets $(j \in J(i) \Leftrightarrow i \in I(j))$

Weighted Extended B-Splines

$$B_i = \frac{w}{w(x_i)}\left(b_i + \sum_{j\in J(i)} e_{i,j}\, b_j\right) = w \sum \tilde{e}_{i,j}\, b_j, \qquad i \in I$$

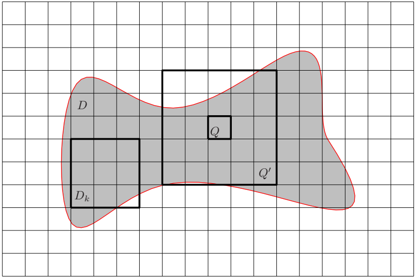




- piecewise polynomial, coordinate degree n
- local support, width $\asymp h$
- uniformly stable
- optimal approximation order

Definitions

$$D_k = \text{supp} \quad b_k$$
$$Q' = \bigcup_{D_k \cap Q \neq \emptyset} D_k$$



Sobolevnorm: $\|u\|_\ell^2 = \sum_{|\alpha| \leq \ell} \int_D |\, \partial^\alpha u \,|^2$

P_n : projection onto polynomials of degree $\leq n$

$P_h v = \sum_k \left(\int_{D_k} \lambda_k v \right) b_k$: projection onto splines

Basic Theorems

Bramble-Hilbert

$$\|v - P_n v\|_{0,Q} \lesssim h^{n+1} \|v\|_{n+1,Q}$$

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Marsden

$$p(x) = \sum_{k \sim D} q(k) b_k(x), \qquad \text{degree } p, q \leq n$$

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$$\|\tilde{v}\|_{\ell,\mathbb{R}^d} \lesssim \|v\|_{\ell,D}$$

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Regularity of Quotients (Höllig, Reif, Wipper)

$$\|v\|_{\ell,D} \lesssim \|wv\|_{\ell+1,D}$$

Jackson Inequality

$$\|u - u_h\|_{0,D} \lesssim h^{n+1} \|u\|_{n+1,D}, \qquad u_h = \sum_i c_i B_i$$

assumptions: $u = wv$, w and v smooth, $w > 0$, $w = 0$ and $\partial_\perp w \neq 0$ on ∂D

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$$\lesssim_{(Q)} (\max_Q |w|) \|\tilde{v} - P_n\tilde{v}\|_{0,Q'} \lesssim_{(B)} h^{n+1} \|\tilde{v}\|_{n+1,Q'}$$

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square, sum over Q

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square, sum over Q

$$\|w\tilde{v} - wP_h\tilde{v}\|^2_{0,D} \lesssim h^{2n+2}\,\|\tilde{v}\|^2_{n+1,\mathbb{R}^d}$$

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square, sum over Q

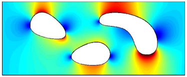
$$\|w\tilde{v} - wP_h\tilde{v}\|^2_{0,D} \lesssim h^{2n+2} \|\tilde{v}\|^2_{n+1,\mathbb{R}^d}$$

$$\lesssim_{(E)} h^{2n+2} \|v\|^2_{n+1,D} \lesssim_{(R)} h^{2n+2} \|u\|^2_{n+2,D}$$

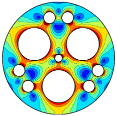
Results for Web-Splines

- optimal approximation order
- uniform stability
- bounds for derivatives
- standard conditioning of Ritz-Galerkin matrices
- multigrid convergence
- error estimates for hierarchical refinement

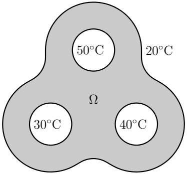
Applications



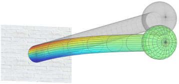
potential flow



plane strain / plane stress

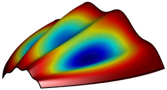


heat conduction



three dimensional elasticity

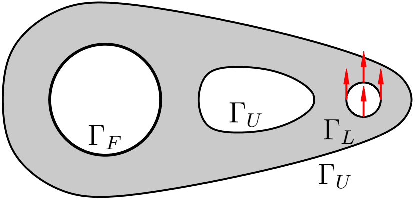
water waves



deformation of shells

Linear Elasticity (Crank Arm) : Convergence Rates

$$
\begin{aligned}
-\operatorname{div}\sigma(u) &= f && \text{in } D,\\
u &= 0 && \text{on } \Gamma_F,\\
\sigma(u)n &= g && \text{on } \Gamma_L
\end{aligned}
$$

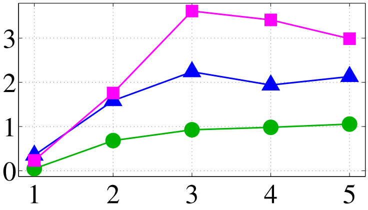


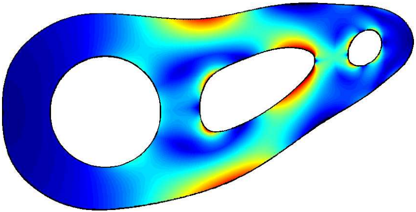
Weak formulation: $\forall v \in \left(H^1_{\Gamma_F}(D)\right)^3$

$$\int_D \sigma(u) : \varepsilon(v) = \int_D f \cdot v + \int_{\Gamma_L} g \cdot v$$

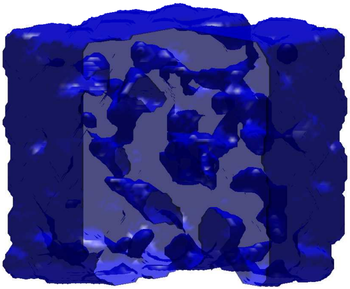
$h_i = h_0 \cdot 2^{-i},\ i \in \{0, \dots, 5\}$
$n \in \{2, 3, 4\}$
$w(x) = x_1^2 + x_2^2 - r_F^2$

Rate of residual norm $\| \operatorname{div} \sigma(u) - f \|_0$

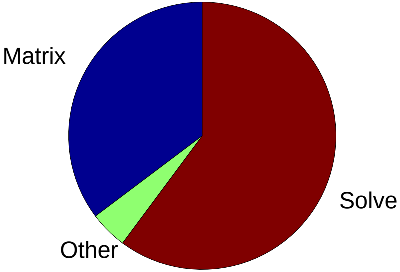




Performance for 3-Dimensional Random Domain

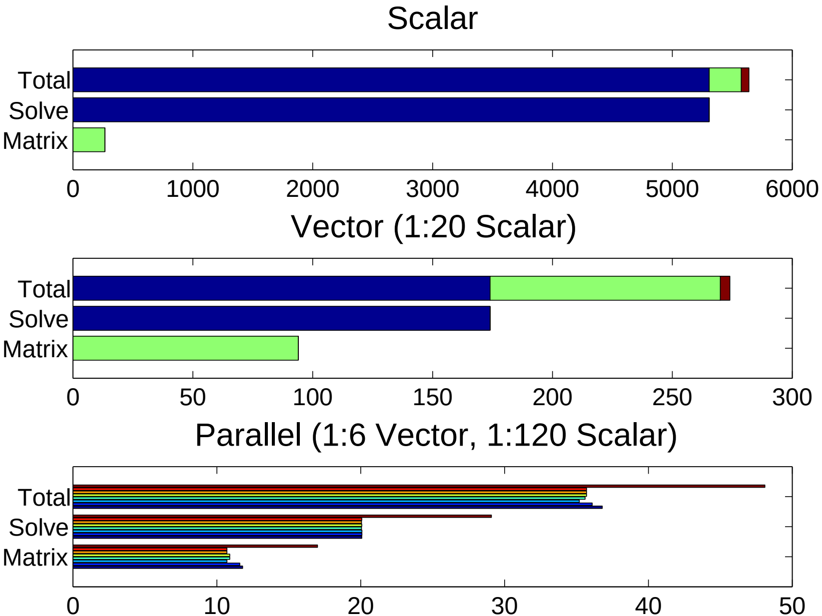


Total time 48.2s (265s)



Problem type:	Dirichlet
Discretization:	192,100,033 (577^3) unknowns, 7 grids
Solver:	Dynamic-Multigrid-Jacobi (10 smoothings, $\omega = 0.88$)
Residual:	5.462E-09 ($<$ 1E-8)
Iterations:	12.1289
Machine:	NEC-SX8, 8 CPUs (1 Node), 103 GB Memory

Time Comparison



Matrix Assembly

```
...

% loop over grid cells
for kx = 1:lgx; for ky = 1:lgy;

   % map evaluation points to grid cell
   pxk = px + gx(kx); pyk = py + gy(ky);

   % values of weight and force function
   W = weight(pxk,pyk);
   f = feval(fct,pxk,pyk);

   % loop over b-splines-i
   for ix=kx:kx+n; for iy=ky:ky+n;
      % right-hand side
      val = f.(W.w).*B{n+1+kx-ix,n+1+ky-iy}.b;
      F(ix,iy) = F(ix,iy) + GRID.c{kx,ky}(:)'*val(:);

      % loop over b-splines-j
      for jx=kx:kx+n; for jy=ky:ky+n;
         val = A(W, B{n+1+kx-ix,n+1+ky-iy}, ...
                 B{n+1+kx-jx,n+1+ky-jy});
         G(ix,iy,jx-ix+n+1,jy-iy+n+1) = ...
            G(ix,iy,jx-ix+n+1,jy-iy+n+1) + ...
            GRID.c{kx,ky}(:)'*val(:);
      end; end;
   end; end;
end; end;

...
```

Program Demo

- Model problem: $-\Delta u = f$
- Domains: unit disc with random holes or random domain
- Programming language: FORTRAN 90
- MATLAB interface for visualization

disc, f constant	2 holes	4 holes
disc, f exponential function	2 holes	4 holes
random domain	f constant	f exponential function

Advantages of the WEB-Method

- No grid generation
- Natural integration in CAD/CAM-systems based on tensor product B-splines
- Simple implementation and short computing times
- Approximations of arbitrary order by appropriate choice of the degree of the basis functions
- Low dimensional approximation spaces, one coefficient per node
- Exact fulfillment of boundary conditions
- Well suited for multigrid methods and hierarchical refinement
- Natural parallelization of algorithms

Current Projects

- MIND: Multiple Integration over NURBS Domains (J. Hörner)
- Shells (M. Boßle)
- Singularities (C. Apprich)
- Adaptive Approximation (M. Mustahsan)
- Boundary Integration (M. Tränkle)
- Multigrid Algorithms (E. Kaygisiz, A. Kniehl)
- Parametric WEB-Splines
- Discontinous Coefficients (A. Keller)

AND

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