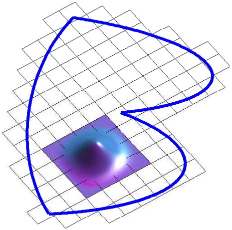
Parallel Finite Element Methods with Weighted Linear B-Splines

K. Höllig, J. Hörner, and M. Pfeil



http://www.web-spline.de

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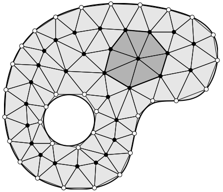
Cooperation partners: Prof. Dr. U. Reif, Dr. J. Wipper

Supported by: TLB, HLRS

Ritz Galerkin Method

Approximation with finite elements

$$u_h = \sum_{k \in K} c_k B_k \in V_h \subset H$$



Weak and variational form

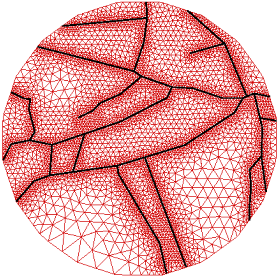
$$\begin{aligned} a(u_h, v_h) &= \lambda(v_h) \quad \forall v_h \in V_h \\ \iff \quad Q(u_h) &= \frac{1}{2} a(u_h, u_h) - \lambda(v_h) \quad \to \quad \min \quad \text{on } V_h \end{aligned}$$

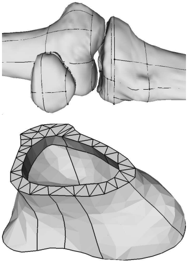
Linear System $GC = F$ for coefficients

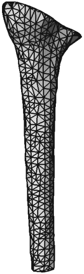
$$g_{j,k} = a(B_j, B_k), \qquad f_j = \lambda(B_j)$$

Standard Finite Elements

ART Triangulations (A. Fuchs)

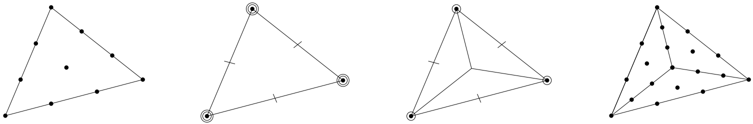






Element types for triangulations

Lagrange	Argyris	Clough-Tocher	Lagrange



$(3, 0, 10)$	$(5, 1, 21)$	$(3, 1, 12)$	$(3, 0, 20)$

(degree, smoothness, dimension)

Splines: Some Historic Steps

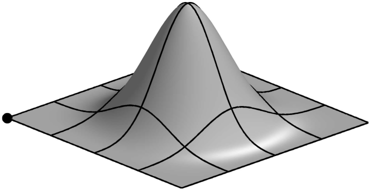
1947	Curry and Schoenberg (B-Splines)
1959/66	de Casteljau, Bézier (Automated Design and Manufacturing)
1967	Ahlberg, Nilson, and Walsh (first book)
197*	de Boor, Schumaker (Numerical Analysis, Approximation)
1979	Dahmen, Michelli (Multivariate Splines)
1980	Oslo algorithm: Cohen, Lyche, and Riesenfeld (Computer Graphics and Image Processing)
1980	Böhm (Computer Aided Geometric Design)

Λ splines, β-splines, ν-splines, ω-splines, τ-splines, A-splines, ARMA splines, B-splines, Bernoulli splines, BM-splines, Box-splines, box-splines, cardinal splines, Catmull-Rom splines, D^m-splines, Dirichlet splines, discrete splines, E-splines, elliptic splines, exponential Euler splines, exponential box splines, fundamental splines, g-splines, Gibbs-Wilbraham splines, $H^{m,p}$-splines, harmonic splines, Helix splines, Hermite splines, Hermite-Birkhoff splines, histosplines, hyperbolic splines, Inf-convolution splines, K-splines, L-monosplines, L-splines, Lagrange splines, LB-splines, Legendre splines, $Lg-$splines, M-splines, metaharmonic splines, minimal-energy splines, monosplines, natural splines, NBV-splines, NURBS, ODR splines, PDL_g splines, perfect splines, PL_{-g} splines, $pLg-$splines, polyharmonic splines, Powell-Sabin splines, pseudo splines,

Multivariate Splines

Multivariate B-splines

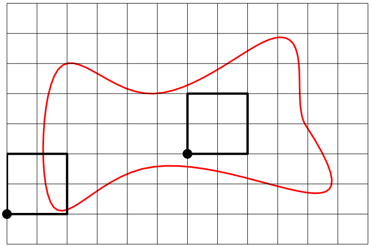
$$b_k(x_1, \ldots, x_m) = \prod_{\nu=1}^{m} b_{k_\nu}(x_\nu)$$



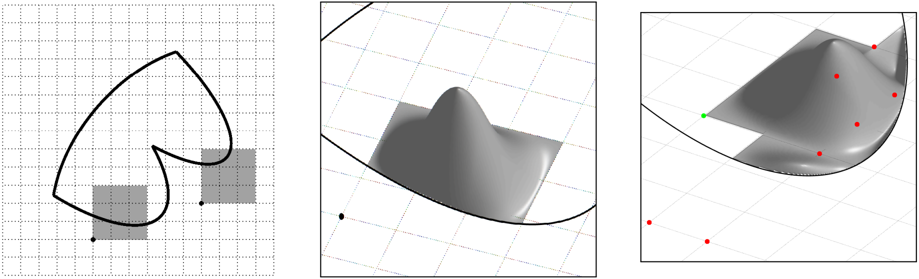
Linear combinations of relevant B-splines

$$f(x) = \sum_{k \in K} c_k b_k$$

$$K : k \sim D$$



B-Splines as Finite Elements



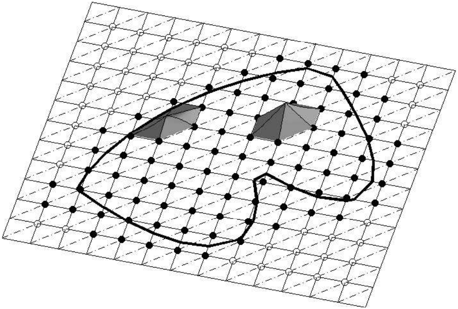
homogeneous boundary condition via weight function

$$b_k \longrightarrow \quad w\, b_k$$

stability via extension

$$b_i \longrightarrow \quad b_i + \sum_{j \in J(i)} e_{i,j} b_j$$

Finite Element Basis



domain

$D \approx D^h \; : \quad w^h(x) > 0$

weighted B-splines

$w^h b^h_k \,, \quad k_\nu \in \mathbb{Z}$

approximation

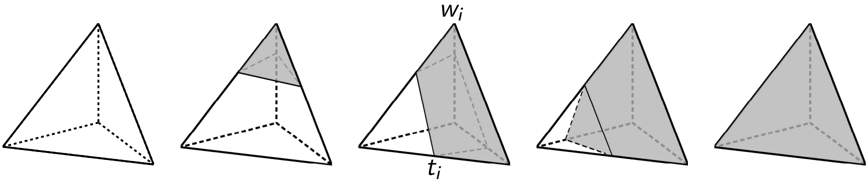
$$u_h = \left(\sum_k w_k b^h_k \right) \left(\sum_k c_k b^h_k \right)$$

Numerical Integration

Ritz-Galerkin integrals

$$\int_{S \cap D^h} a\left(w^h b_k^h, w^h b_{k'}^h\right) = I\left(w_S^h, k, k'\right)$$

intersection patterns



Poisson bilinear form

I : rational function of w_S^h or t_S^h

(automatically generated, $6 * 16 * 10$ cases)

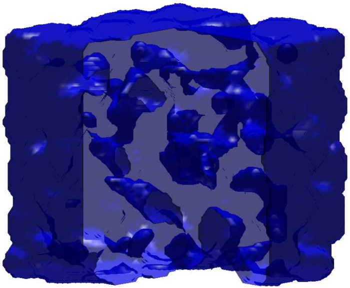
Simplification of Expressions for Ritz-Galerkin Integrals

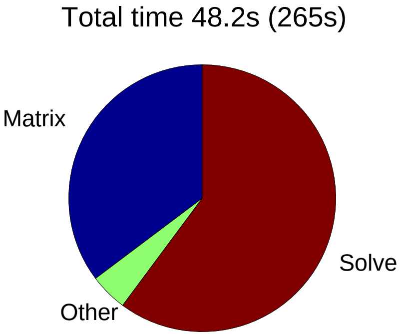
$$\frac{w_{0,0,0}^4 \left(2w_{0,0,0}^2 w_{1,1,0}^2 w_{1,1,1}^3 + 6w_{0,0,0}^4 w_{1,0,0} w_{1,1,1}^2 + \ldots \quad \text{(107 similar terms)}\right)}{\left(w_{0,0,0} - w_{1,0,0}\right)^3 \left(w_{0,0,0} - w_{1,1,0}\right)^2 \left(w_{0,0,0} - w_{1,0,1}\right)^2 \left(w_{0,0,0} - w_{1,1,1}\right)^2}$$

$$\left.\begin{array}{rl} \text{4 substitutions of type} & \tilde{w} = a/(b-c) \\ \text{17 substitutions of type} & \tilde{w} = a \cdot b \\ \text{4 substitutions of type} & \tilde{w} = a/b \end{array}\right\} \rightarrow \quad \text{simplified instruction set}$$

$$20\tilde{w}_5 - 6\tilde{w}_6 - \tilde{w}_7 - 10\tilde{w}_8 - 5\tilde{w}_9 - 10\tilde{w}_{10} - \tilde{w}_{11} - 5\tilde{w}_{12} + 10\tilde{w}_{13} + 2\tilde{w}_{14} + 2\tilde{w}_{15}$$
$$-2\tilde{w}_{16} - 8\tilde{w}_{17} + 4\tilde{w}_{18} - 2\tilde{w}_{19} + \tilde{w}_{20} + 2\tilde{w}_{21} + 4\tilde{w}_{22} + 2\tilde{w}_{23} + \tilde{w}_{24} + 2\tilde{w}_{25}$$

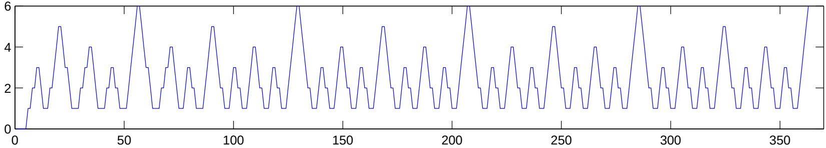
Example: 3-Dimensional Random Domain





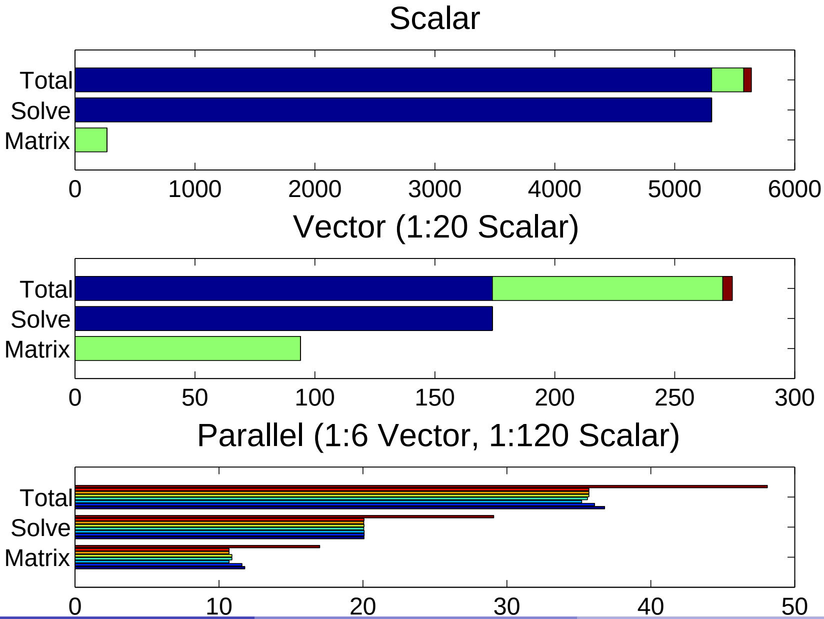
Problem type:	Dirichlet
Discretization:	192,100,033 (577^3) unknowns, 7 grids
Solver:	Dynamic-Multigrid-Jacobi (10 smoothings, $\omega = 0.88$)
Residual:	5.462E-09 ($<$ 1E-8)
Iterations:	12.1289
Machine:	NEC-SX8, 8 CPUs (1 Node), 103 GB Memory

Example: Multigrid Levels



		Grid 0	Grid 1	Grid 2	Grid 3	Grid 4	Grid 5	Grid 6
calls		5	112	106	75	37	19	9
sopt	s	0.0	1.5	10.6	58.0	226.5	924.7	3729.2
	%	0.0	0.0	0.2	1.0	4.0	16.3	65.6
hopt	s	0.0	0.2	0.8	2.6	9.9	35.7	114.3
	%	0.0	0.1	0.3	0.9	3.6	13.1	41.7
hopt +	s	0.0	0.1	0.1	0.3	1.2	4.5	15.2
omp	%	0.0	0.3	0.3	0.8	2.5	9.4	31.6

Example: Time Comparision



Example: Ftrace-Statistics (1 CPU)

PROG_UNIT	TIME	MFLOPS	V.RATIO	V.LEN
TOTAL	273.787	3553.8	97.74	165.6
MATRIX	96.207	1508.5	90.00	133.1
MG_SOLVE	174.356	4733.5	99.21	172.6
MATRIX_CELLTYPES	6.107	0.0	56.52	183.8
CELL_DATA_255	16.028	7208.2	99.13	156.5
⋮				
CELL_DATA_066	0.000	55.5	49.45	32.1
MG_SMOOTH_6	114.302	5055.4	99.37	192.5
MG_SMOOTH_5	35.746	4290.9	99.05	144.7
⋮				
MG_SMOOTH_0	0.003	563.8	84.53	11.0

Example: Ftrace-Statistics (8 CPU)

PROG_UNIT	TIME	MFLOPS	V.RATIO	V.LEN
TOTAL	264.565	3617.5	97.75	165.8
MATRIX	94.219	1371.6	89.64	133.0
MG_SOLVE	169.636	4865.2	99.21	172.6

PROCESS	MATRIX	MG_SOLVE	MG_S_6
micro1	16.984	29.096	15.221
micro2	10.719	20.084	14.372
micro3	10.709	20.059	14.366
micro4	10.882	20.075	14.380
micro5	10.859	20.060	14.349
micro6	10.745	20.092	14.380
micro7	11.553	20.088	14.376
micro8	11.768	20.083	14.374
sum	94.219	169.636	115.817

Advantages of the WEB-Method

- No grid generation
- Natural integration in CAD/CAM-systems based on tensor product B-splines
- Simple implementation and short computing times
- Approximations of arbitrary order of accuracy by appropriate choice of the degree of the basis functions
- Low dimensional approximation spaces
- Exact fulfillment of boundary conditions
- Well suited for multigrid methods and hierarchical refinement
- Natural parallelization of algorithms