The Web-Method

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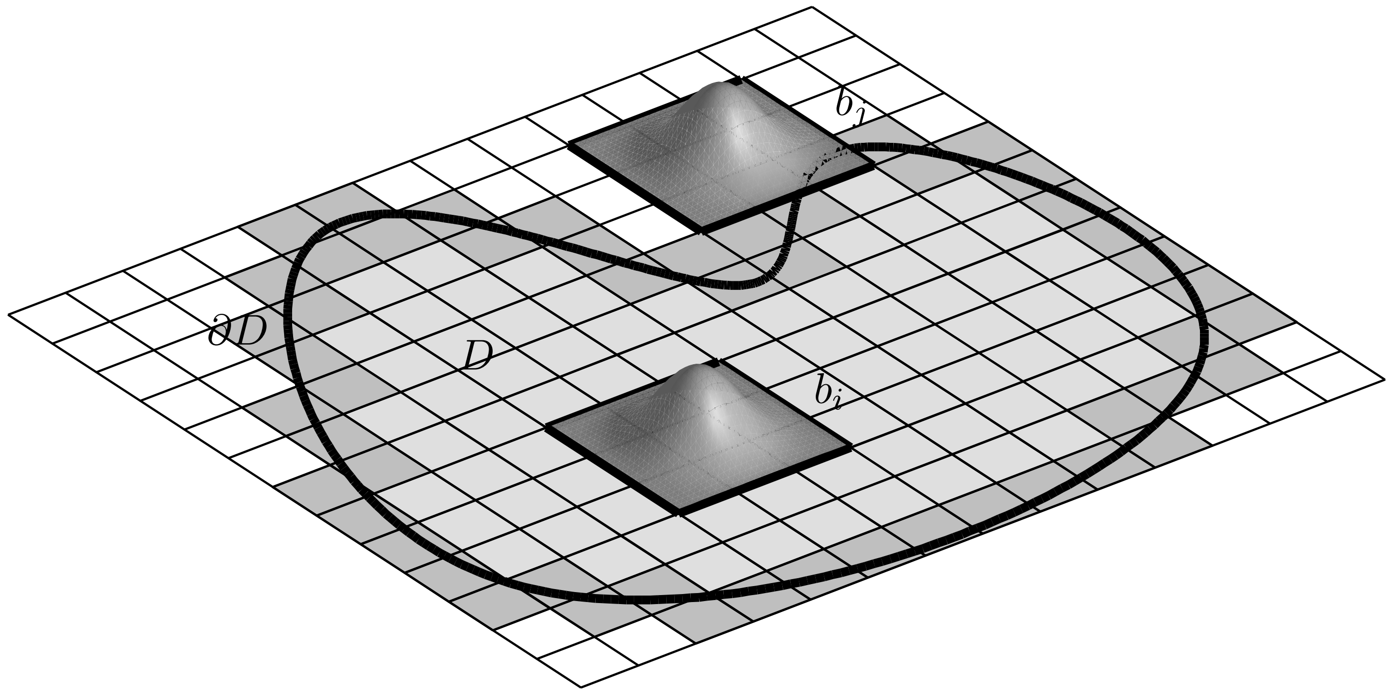
http://www.web-spline.de

Splines on bounded Domains

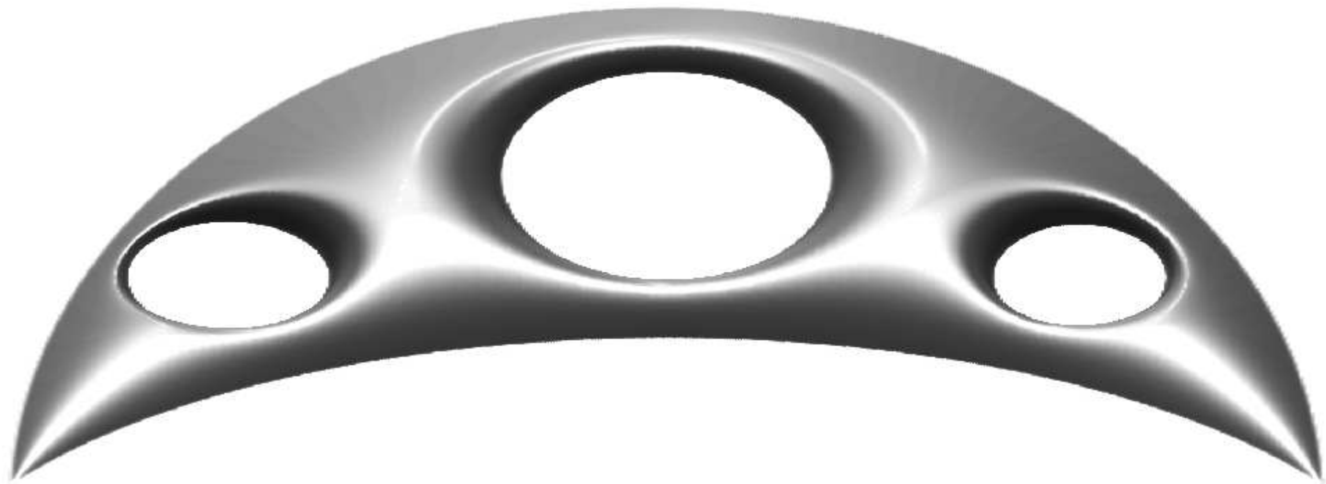


$$\mathbb{B}_h = \operatorname*{span}_K b_k\,,\; K = I \cup J$$

boundary conditions via weight function

stability via extension of inner B-splines

Weight function



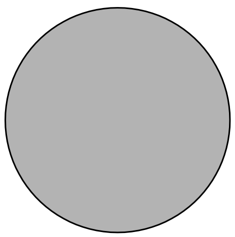
essential boundary conditions $w_{|_\mathcal{D}} > 0\,,\ w_{|_\Gamma} = 0, \quad \Gamma \subseteq \partial\mathcal{D}$

construction:

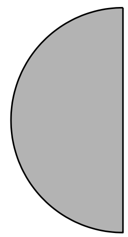
- explicit formulas
- Rvachev's Boolean expressions
- numerical distance functions
- Reif's blending technique

Rvachev's R-Functions

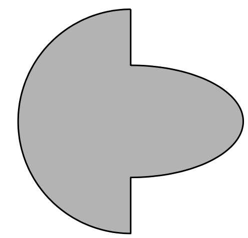
$w_1 = 1 - x_1^2 - x_2^2$



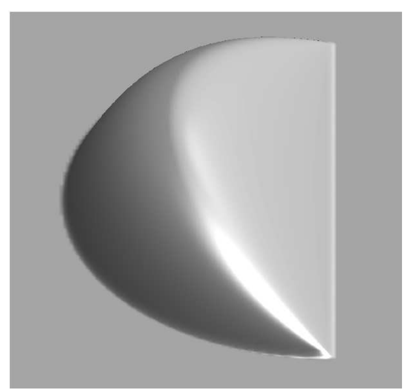
$w_2 = -x_1$



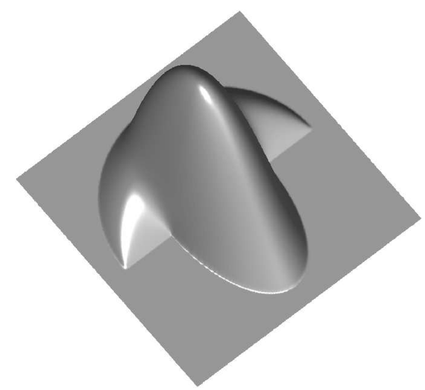
$w_3 = 1 - x_1^2 - 4x_2^2$



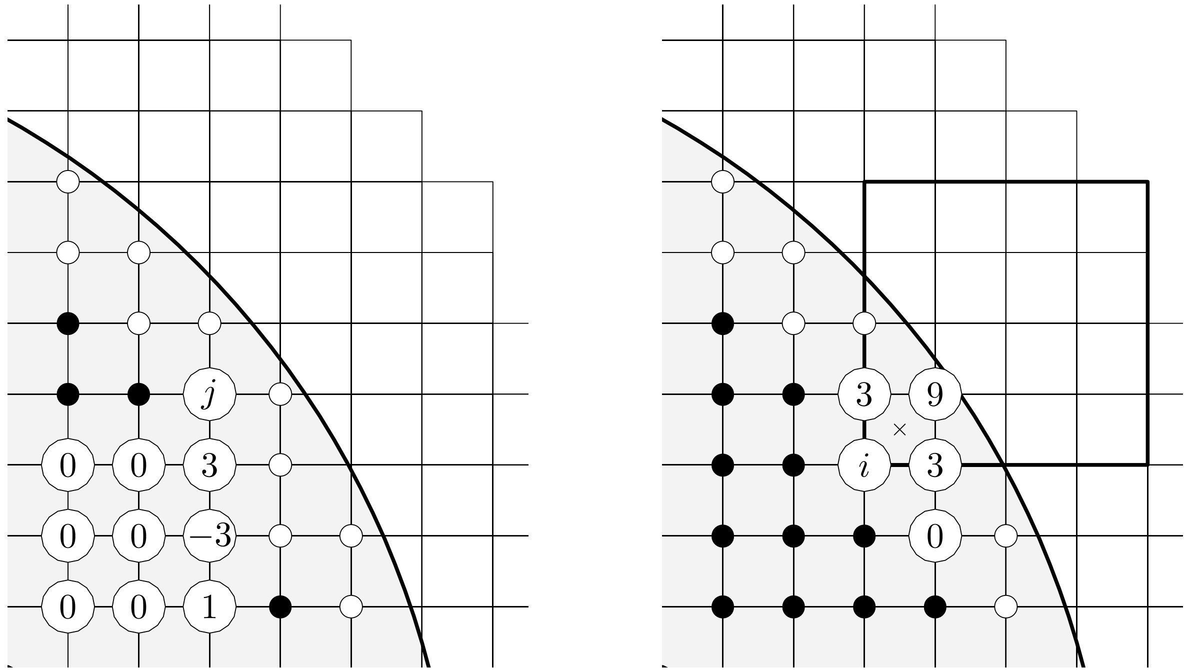
$w_\cap = w_1 + w_2 - \sqrt{w_1^2 + w_2^2}$



$w_\cup = w_\cap + w_3 + \sqrt{w_\cap^2 + w_3^2}$



Extension



$I(j)$: nearest $(n+1)^m$-array of inner indices

$J(i)$: complementary sets $(i \in I(j) \Leftrightarrow j \in J(i))$

$e_{i,j}$: values of Lagrange polynomials (uniform grid) or de Boor functionals (nonuniform grid)

[Nonuniform] Weighted Extended B-Splines

$$B_i = \frac{w}{w(x_i)} \left(b_i + \sum_{j \in J(i)} e_{i,j} b_j \right)$$

properties:

- local support: $e_{i,j} = 0$ for $||i - j|| \gtrsim 1$
- stability: $|| \sum c_i B_i ||_0 \asymp h^{m/2} ||C||$
- approximation order: $||u - P_h u||_\ell \lesssim h^{n+1-\ell} ||u||_{n+1}$

Ritz-Galerkin Approximation

H : Hilbert space, incorporating homogeneous boundary conditions

a : elliptic bilinear form

λ : linear functional

weak solution:

$$a(u,v) = \lambda(v)\,,\; v \in H$$

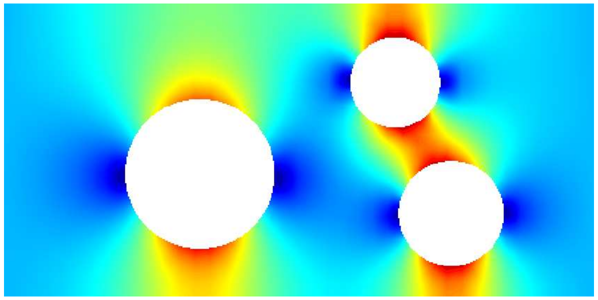
finite element approximation:

$$a(u_h, B_i) = \lambda(B_i)\,,\; i \in I$$

error estimate:

$$||u - u_h||_H \lesssim \inf_C \left\| u - \sum c_i B_i \right\|_H$$

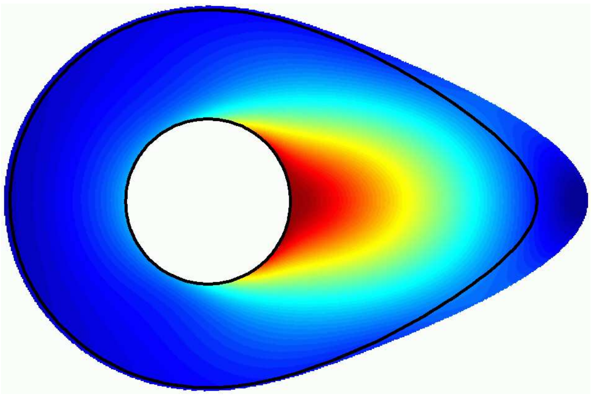
Examples of boundary value problems



incompressible flow

$-\Delta u = 0$ in D

$\frac{\partial u}{\partial n} = -v$ on Γ_i, $\quad \frac{\partial u}{\partial n} = v$ on Γ_o, $\quad \frac{\partial u}{\partial n} = 0$ on Γ_b



linear elasticity

$-\mathrm{div}\ \sigma(u) = f$ in D

$u = 0$ on Γ_0, $\quad \sigma(u)n = 0$ on Γ_1

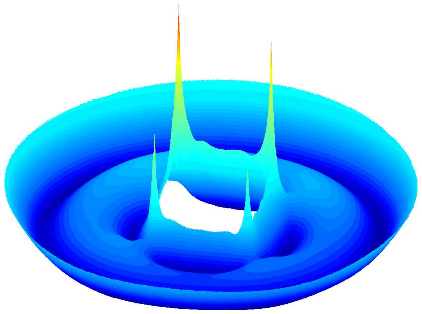
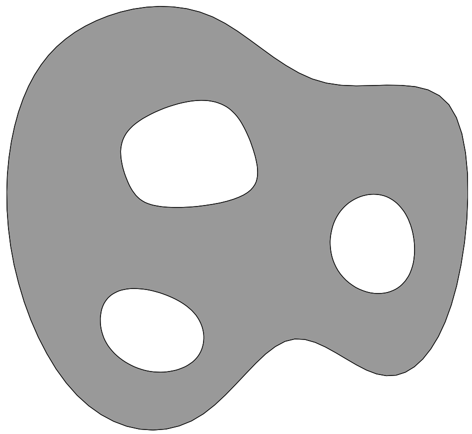


plate problem

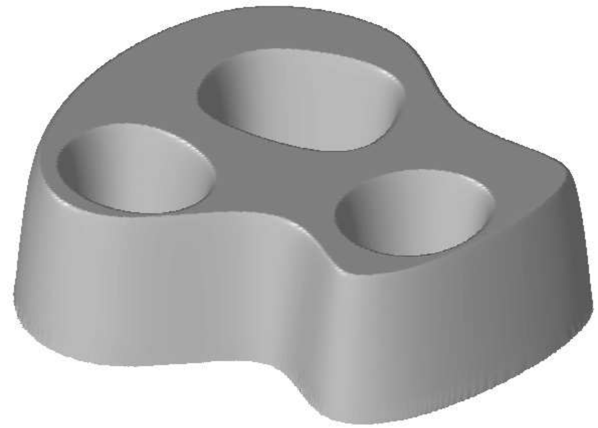
$\Delta^2 u = f$ in D

$u = \frac{\partial}{\partial n} u = 0$ on ∂D

WEB-Splines on NURBS Domains

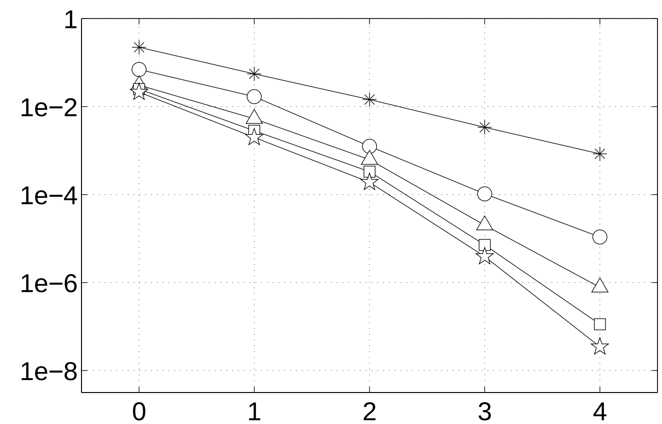


domain



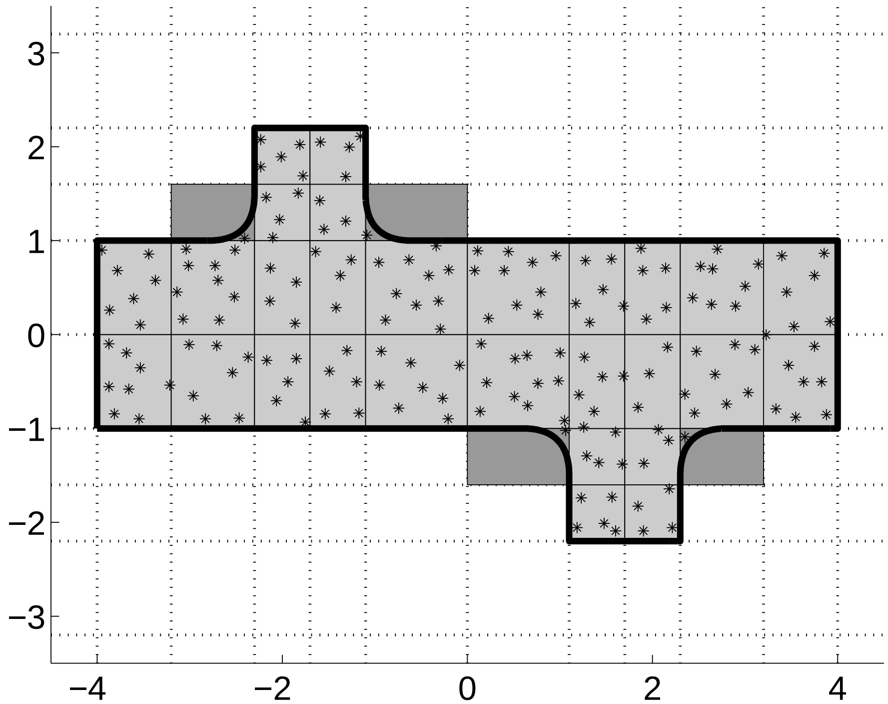
weight function

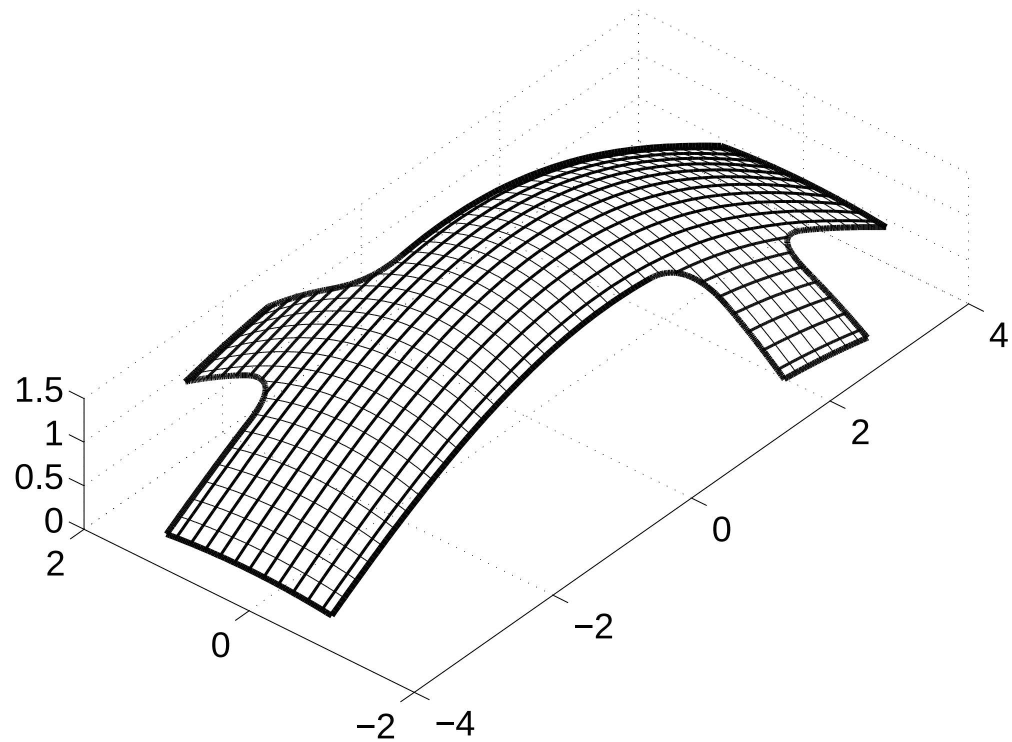
solution to $-\Delta u = 1$



L_2 error

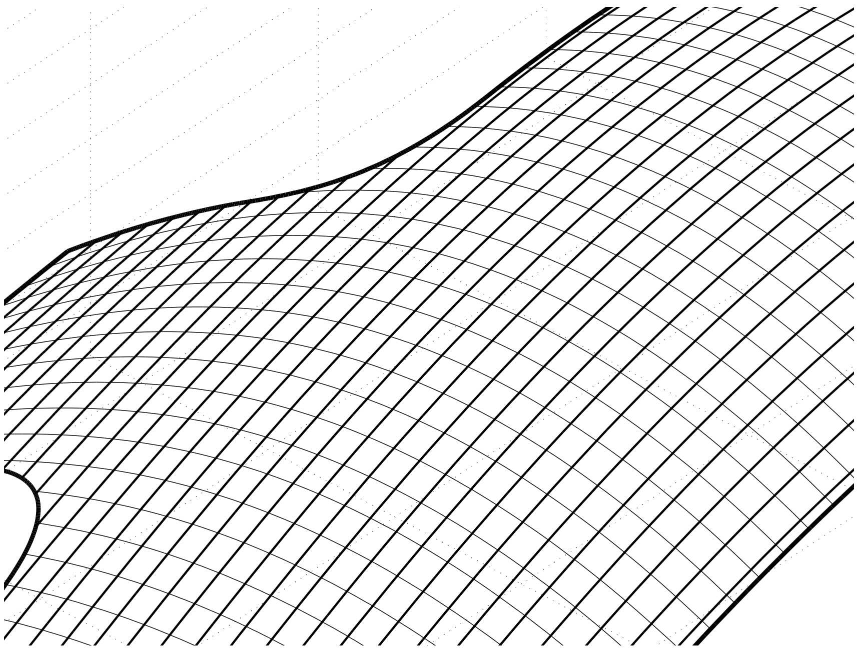
Reverse Engineering





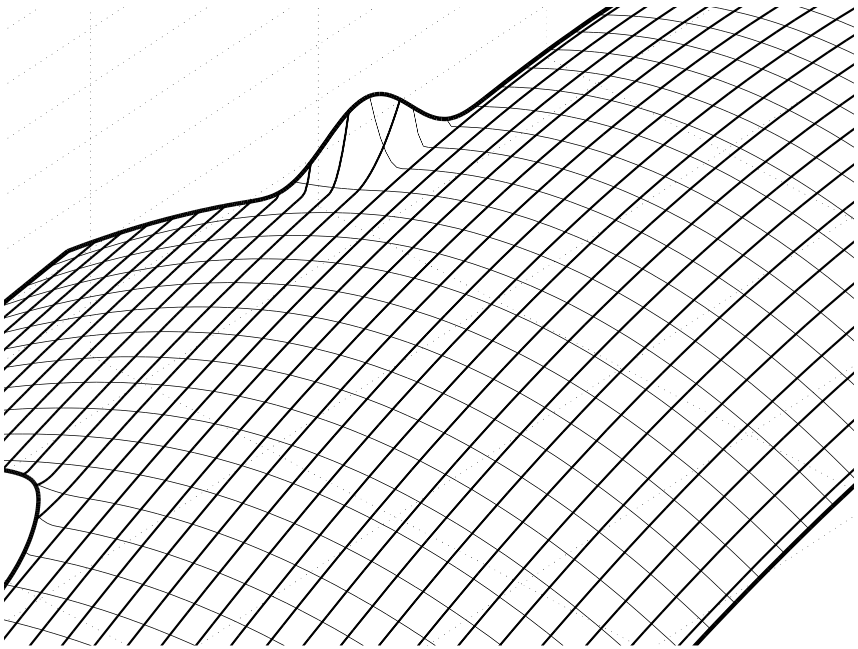
Scattered data on a trimmed domain, sampled from a smooth function

Reverse Engineering



Smooth solution by web-splines

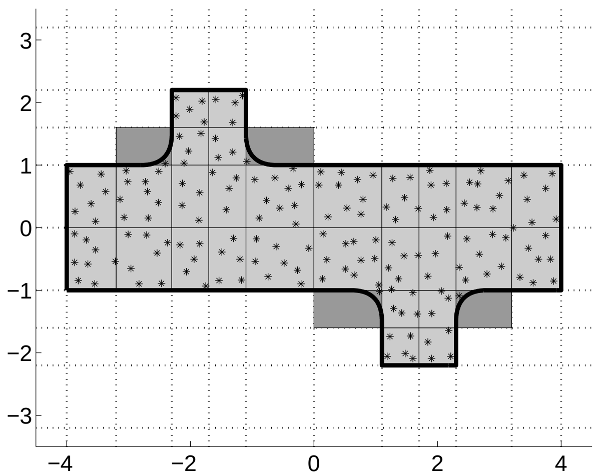
$\|q_{\text{web}} - f\|_\infty \approx 2.2\text{e-}4$

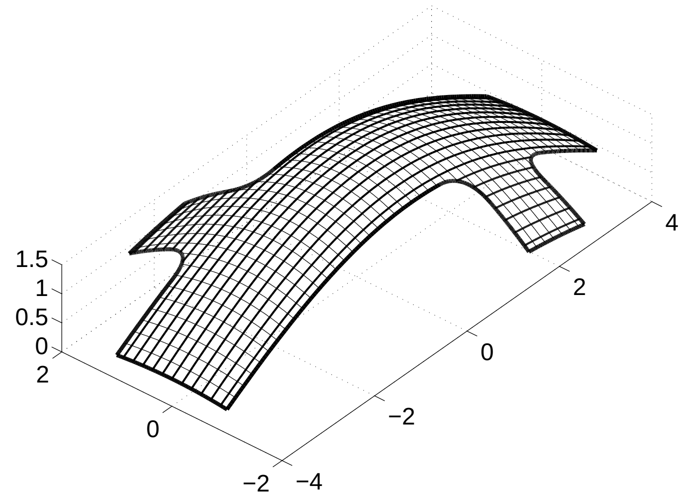


Artifacts by standard NURBS

$\|q_{\text{std}} - f\|_\infty \approx 2.8\text{e-}1$

Reverse Engineering





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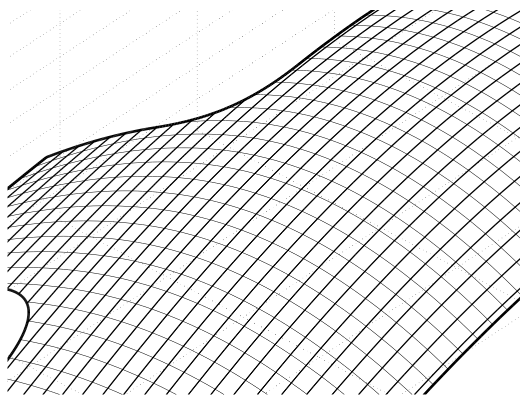

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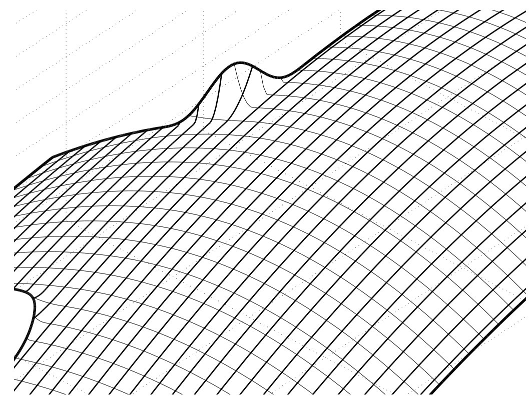

Artifacts by standard NURBS

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Properties of Web-Splines

- regular grid
- local support
- arbitrary degree and smoothness
- uniform stability
- high accuracy with few parameters
- exact fulfilment of boundary conditions
- adaptive refinement via local error estimators
- multigrid techniques

Multigrid history

- Fedorenko 1964: multigrid algorithm for standard five-point finite difference discretization of Poisson's equation on a square, $\mathcal{O}(n)$
- Bakvahlov 1966: generalization to difference discretizations of general elliptic pde's with variable smooth coefficients
- Brandt 1973: first practical results, important paper in 1977 outlining the main priciples and practical utility of multigrid
- Bank, Dupont 1981: multigrid for finite elements
- Ries, Trottenberg, Winter 1983: multigrid reduction algorithm
- Braess, Hackbusch 1983: V-cycle convergence proof

Model Problem

Poisson equation $\quad -\Delta u = f, \quad u \in H_0^1$

$$a(u,v) = \int_D \operatorname{grad} u \operatorname{grad} v, \quad \lambda(v) = \int_D fv$$

Ritz-Galerkin Approximation

$$GU = F \quad \Longleftrightarrow \quad a\left(\sum u_i B_i, B_k\right) = \lambda(B_k)$$

regularity, approximation, and stability

$$\|u\|_2 \preceq \|f\|_0$$

$$\|u - u_h\|_\ell \preceq h^{k-\ell}\|u\|_k, \quad \ell \le k \le n+1$$

$$\operatorname{cond} G_h \asymp h^{-2}$$

Richardson Iteration

one step

$$U \leftarrow S(U,F) = U - \gamma^{-1}(GU - F)$$

with $\gamma = \|G\|_\infty = \max_{k\in I} \sum_{i\in I} |g_{k,i}|$

error reduction of α steps $U \to V$

$$V - U_* = (E - \gamma^{-1}G)^\alpha (U - U_*)$$

where $GU_* = F$

eigenvalues $(1 - \lambda_i/\gamma)^\alpha \in [0,1)$, slow convergence

Quasi-Interpolation

projection of coarse grid web-splines

$$P_h \sum_{\ell \in \widetilde{I}} \tilde{u}_\ell \widetilde{B}_\ell = \sum_{i \in I} u_i B_i, \quad U = P\widetilde{U}$$

with $P_h f = \sum_{i \in I} \langle \Lambda_i, f \rangle_0 \, B_i$

uniform boundedness

$$\|P_h f\|_0 \preceq \|f\|_0$$

sparse matrix P, explicitly defined via subdivision

Multigrid algorithm

one step

$$U \to W = \mathcal{M}(U, F, h)\ ,$$

which improves an approximation $U \approx U_* = G^{-1}F$

$$
\begin{aligned}
& V = S^{\alpha}(U, F) \\
& R = GV - F \\
& \widetilde{R} = P^t R \\
& \textbf{if } 2h = h_{\mathsf{max}} \\
& \qquad \widetilde{W} = \widetilde{G}^{-1}\widetilde{R} \\
& \textbf{else} \\
& \qquad \widetilde{W} = \mathcal{M}^{\beta}(0, \widetilde{R}, 2h) \\
& \textbf{end} \\
& W = V - P\widetilde{W}
\end{aligned}
$$

Multigrid heuristic

smooth residuum

$$R = GV - F = G(V - U_*) \approx G(P\widetilde{W})$$

related Galerkin matrices

$$\tilde{g}_{k,i} = a(\widetilde{B}_i, \widetilde{B}_k) \approx a(P_h\widetilde{B}_i, P_h\widetilde{B}_k) = \sum_{\alpha,\beta\in I} p_{\alpha,i}\, a(B_\alpha, B_\beta)\, p_{\beta,k}\,,$$

i.e., $\widetilde{G} \approx P^tGP$

approximate correction

$$P^tG(P\widetilde{W}) \approx \widetilde{G}\widetilde{W} = \widetilde{R} = P^tR$$

implies $W = V - P\widetilde{W} \approx V - G^{-1}R = U_*$

Smoothing

after α Richardson steps $U \to V \approx U_*$

$$\|G(V - U_*)\| \preceq \frac{h^{m-2}}{\alpha + 1} \|U - U_*\|$$

Proof: estimate eigenvalues κ_i of

$$G \left(E - \gamma^{-1} G\right)^\alpha$$

in terms of the eigenvalues $0 < \lambda_i$ of G

(i) $\gamma \preceq h^{m-2}$

(ii) $\vartheta_i = \lambda_i / \gamma \in (0, 1]$

(iii) $|\kappa_i| \leq \max_{\vartheta \in [0,1]} \gamma \, \vartheta (1 - \vartheta)^\alpha = \gamma \frac{1}{\alpha + 1} \left(\frac{\alpha}{\alpha + 1}\right)^\alpha$

Correction

$$\|(v-u_*)-\tilde{u}_*\|_0 \preceq h^{2-m}\|r\|_0,$$

with $R = G(V-U_*)$, $\widetilde{G}\widetilde{U}_* = \widetilde{R} = P^t R$

● ● ● We prove this estimate by interpreting $v-u_*$ and $\tilde{u}_*$ as approximations to the solution φ of an auxiliary Poisson problem $-\Delta\varphi = \eta, \quad \varphi_{|\partial D} = 0$.
To this end we define $\eta \in L_2$ by

$$\langle \eta, \psi\rangle_0 = a(v-u_*, P_h\psi), \quad \forall \psi \in L_2\,. \qquad (1)$$

By Bernstein's inequality, $\|P_h\psi\|_1 \preceq h^{-1}\|P_h\psi\|_0$, i.e., the right-hand side of (1) defines indeed a continuous linear functional on L_2. Hence, the Riesz representation theorem is applicable. From the definition (1) of η it is easily deduced that $\langle \eta, B_i\rangle_0 = r_i, \quad \langle \eta, \widetilde{B}_i\rangle_0 = \tilde{r}_i$.
We just set $\psi = B_i = P_h B_i$ for the first identity and note that $a(v-u_*, B_i)$ is the i-th entry of the vector $G(V-U_*)$. The second identity follows from $\tilde{r}_i = \sum_\ell p_{\ell,i} \underbrace{\langle \eta, B_\ell\rangle_0}_{r_\ell} = a(v-u_*, \sum_\ell p_{\ell,i}B_\ell) = a(v-u_*, P_h\widetilde{B}_i)$.

Therefore, $v-u_*$ and $\tilde{u}_*$ are both Ritz-Galerkin approximations to φ. By the standard error estimate for Poisson's equation, $\|(v-u_*)-\tilde{u}_*\|_0 \le \|(v-u_*)-\varphi\|_0 + \|\varphi-\tilde{u}_*\|_0 \preceq h^2\|\eta\|_0$.

This completes the proof since, with $q = P_h\eta$, $\|\eta\|_0^2 = a(v-u_*, P_h\eta) = QG(V-U_*) \le \|Q\|\|R\|$, and $\|Q\| \asymp h^{-m/2}\|q\|_0 \preceq h^{-m/2}\|\eta\|_0, \quad \|R\| \asymp h^{-m/2}\|r\|_0$ by the stability theorem and the boundedness of the quasi-interpolant P_h. ● ● ●

Grid independent convergence

for $\beta = 2$ (w-cycle)

$$\|W - U_*\| \preceq \frac{1}{\alpha + 1} \|U - U_*\|$$

Proof for the two-grid iteration

$$w = v - P_h \tilde{u}_* \Leftrightarrow W = V - P\widetilde{U}_*, \quad \widetilde{U}_* = \widetilde{G}^{-1}\widetilde{R}$$

With the bound for $\|W - U_*\|$,

$$\begin{aligned} \|V - P\widetilde{U}_* - U_*\| &\asymp h^{-m/2} \|P_h(v - u_*) - P_h \tilde{u}_*\|_0 \\ &\preceq h^{-m/2} \|v - u_* - \tilde{u}_*\|_0, \end{aligned}$$

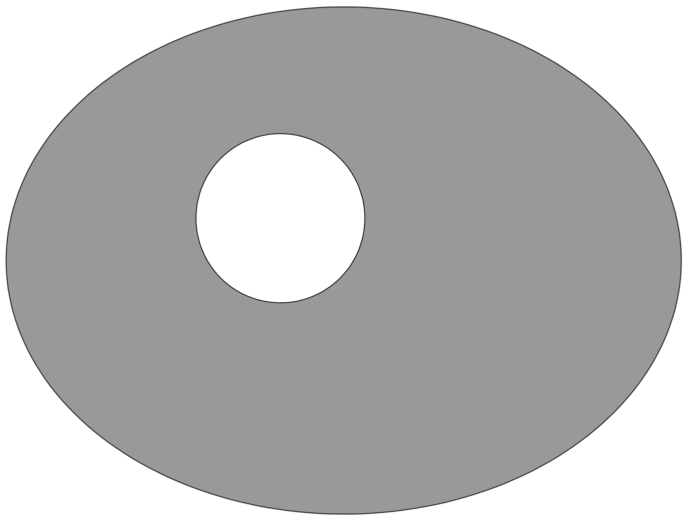
the results about correction and smoothing yield

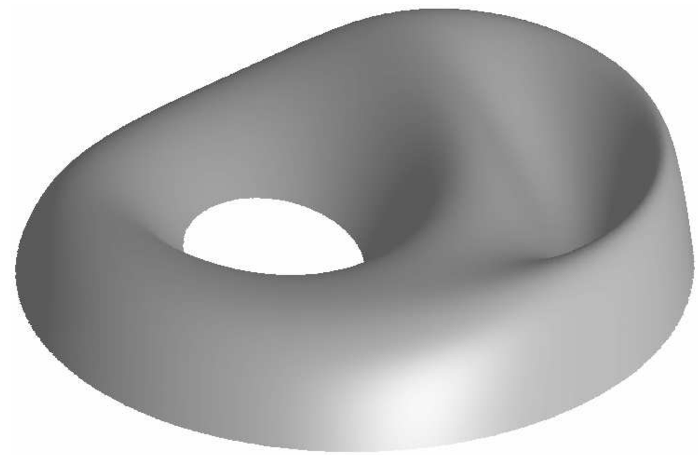
$$\|v - u_* - \tilde{u}_*\|_0 \preceq \frac{h^{m/2}}{\alpha + 1} \|U - U_*\|,$$

using also $\|r\|_0 \asymp h^{m/2} \|R\|$ and $R = G(V - U_*)$.

Multigrid example

Poisson Problem on Ω, bounded by an ellipse and a circle.

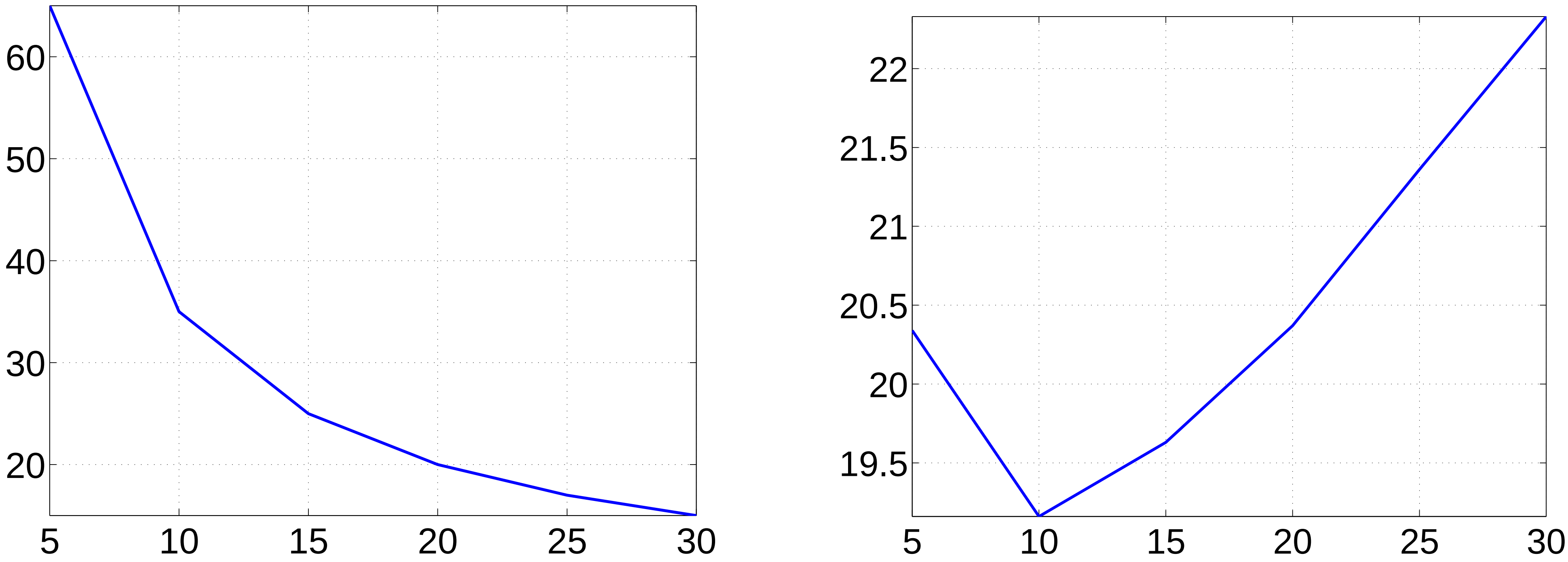




9396 web-splines of degree 3 on finest grid ($h = 2^{-4}$), 56 on coarsest ($h = 2^0$).

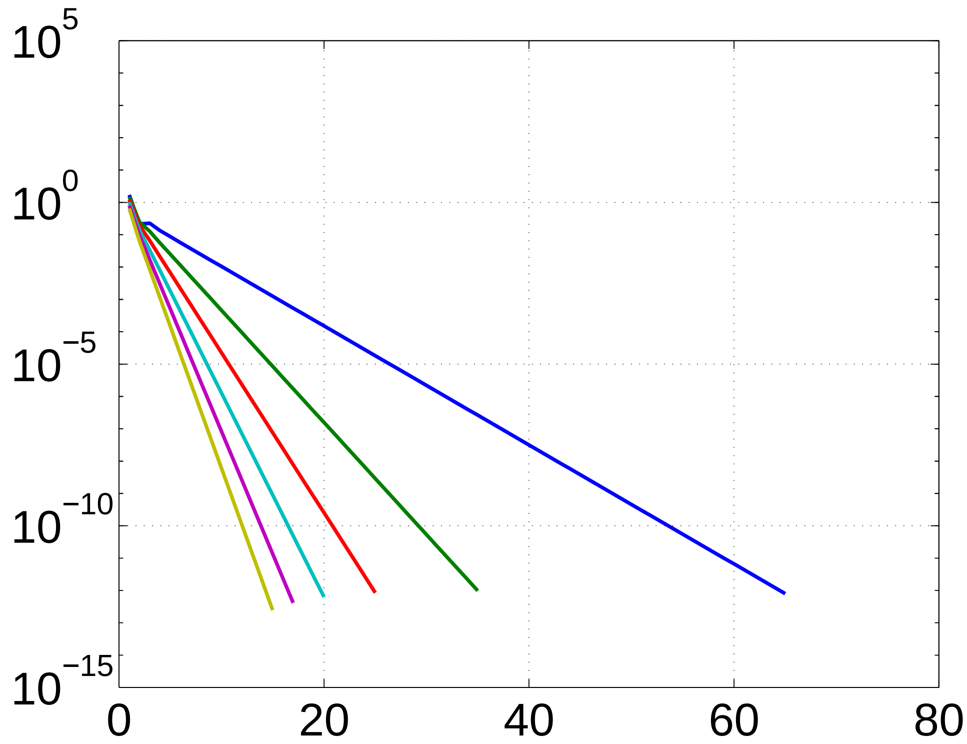
Relative errors $\|u - u_h\|_0$ =4.86e-07, $\|u - u_h\|_1$ =2.29e-05. Using $\beta = 1$ (V-cycle), ssor smoothing and residual norm of 1e-12 as mg stopping criterion.

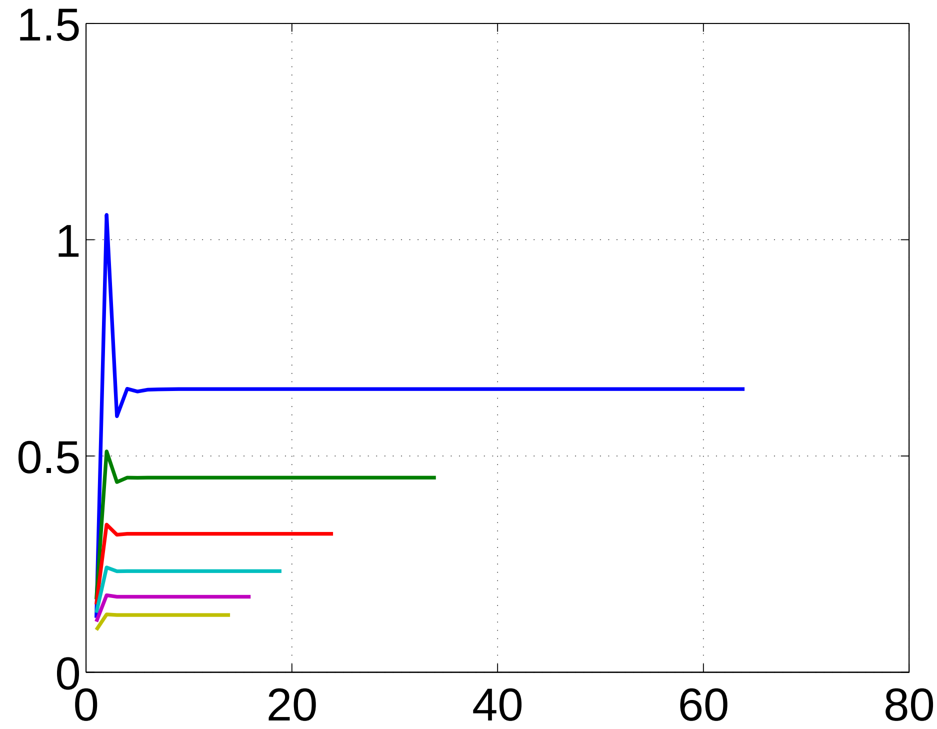
Multigrid example



Number of mg cycles versus number α of smoothing steps (left) and computation time in seconds versus number α of smoothing steps (right)

Multigrid example





Residual norm versus mg cycle number for α=[5:5:30] (left), and rate (right)