Approximation and Modeling with Hierarchical B-Splines

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definitions and B–spline demos $\longrightarrow$

`http://www.siam.org/books/ot132/`

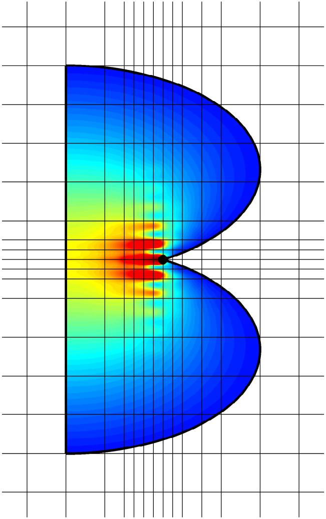
fem program package $\longrightarrow$

`http://www.siam.org/books/fr26/`

classical articles and recent related work:

Forsey, Bartels 1988; Kraft 1998; Sederberg, Zheng, Bakenov, Nasri 2003; Mustahsan 2011; Dokken, Lyche, Petterson 2013; Lehmann 2013

Necessity of Hierarchical Refinement



global effect of knot insertion

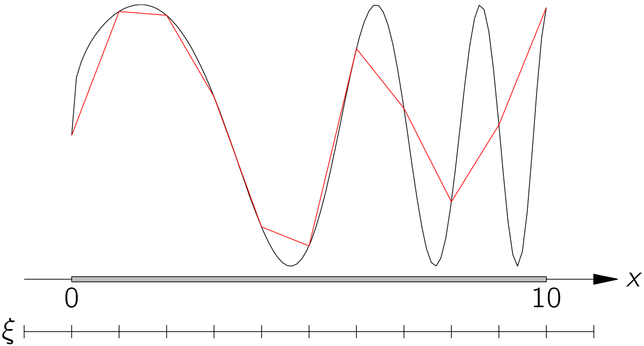
Poisson's equation

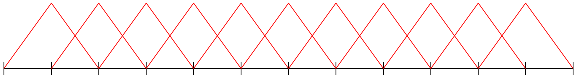
$$-\Delta u = 1 \quad \text{in } D$$
$$u = 0 \quad \text{on } \partial D$$

singularity at reentrant corner

$$|\nabla u| \sim r^{-0.4472}$$

Piecewise Linear Interpolation

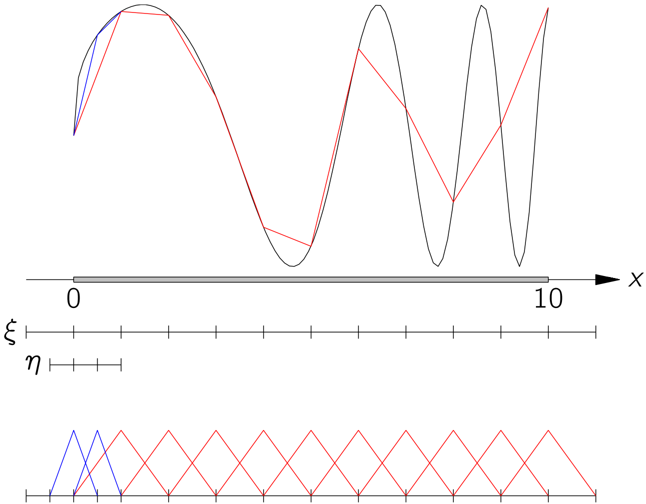




hierarchical basis

$b^1_{0,\xi}, \dots, b^1_{10,\xi}$

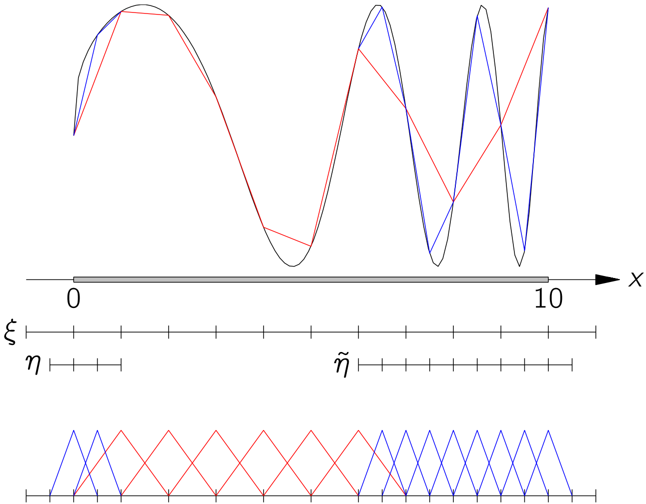
Piecewise Linear Interpolation



hierarchical basis

$$\color{red}{b^1_{1,\xi},\ldots,b^1_{10,\xi}},\quad \color{blue}{b^1_{0,\eta},b^1_{1,\eta}}$$

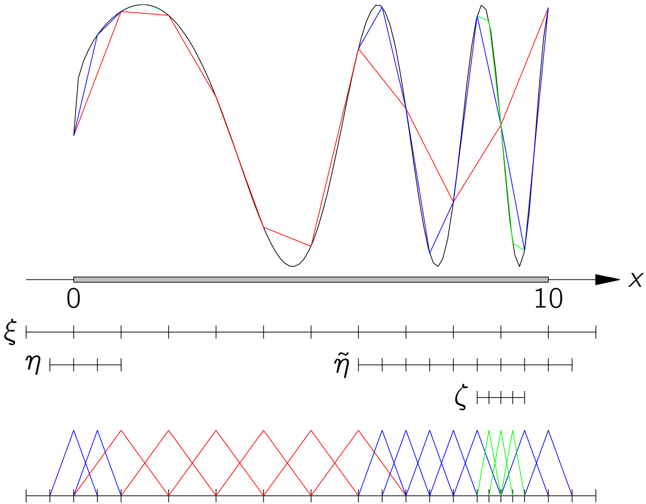
Piecewise Linear Interpolation



hierarchical basis

$b^1_{1,\xi}, \ldots, b^1_{6,\xi}, \quad b^1_{0,\eta}, b^1_{1,\eta}, \quad b^1_{0,\tilde\eta}, \ldots, b^1_{7,\tilde\eta}$

Piecewise Linear Interpolation



hierarchical basis

$b^1_{1,\xi}, \ldots, b^1_{6,\xi}, \quad b^1_{0,\eta}, b^1_{1,\eta}, \quad b^1_{0,\tilde\eta}, \ldots, b^1_{4,\tilde\eta}, b^1_{6,\tilde\eta}, b^1_{7,\tilde\eta}, \quad b^1_{0,\zeta}, b^1_{1,\zeta}, b^1_{2,\zeta}$

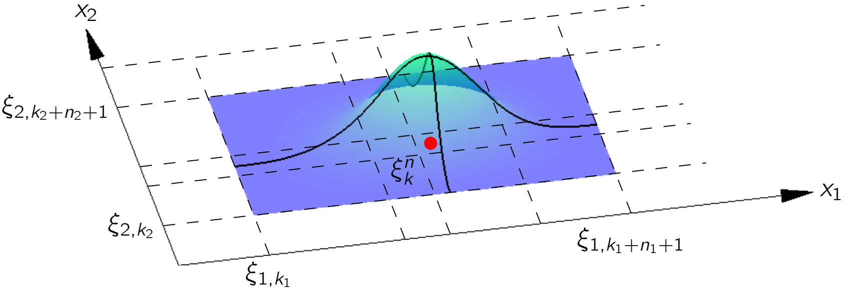
B-Splines

d-variate B-splines of degree $(n_1, \cdots, n_d)$ with respect to knot sequences

$$\xi_\nu : \cdots, \xi_{\nu,0}, \xi_{\nu,1}, \cdots, \quad \nu = 1, \cdots, d,$$

are products of univariate B-splines:

$$b_{k,\xi}^n(x) = \prod_{\nu=1}^{d} b_{k_\nu,\xi_\nu}^{n_\nu}(x_\nu).$$

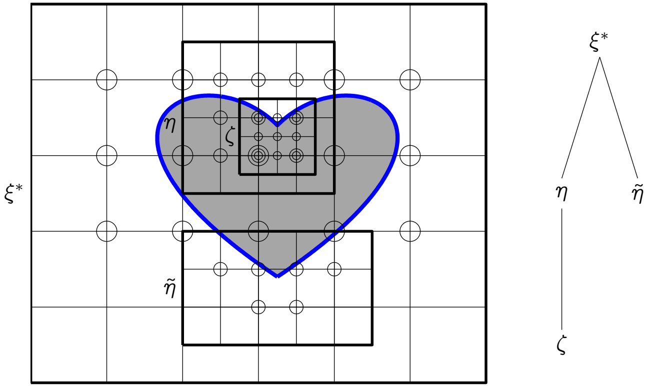


Hierarchical Splines

A hierarchical spline space $S^n_\Xi(D)$ is spanned by B-splines $b^n_{k,\xi}$ from knot sequences ξ, which are nodes of a tree Ξ. A basis for $S^n_\Xi(D)$ consists of those relevant B-splines for D,

$$b^n_{k,\xi}, \quad k \in K_\xi\,,\ \xi \in \Xi\,,$$

which are nonzero at a point in the interior of D outside of the hyperrectangles $[\eta]$ for any of the children η of ξ.



Essential Assumptions

- separation:

$$[\eta] \cap [\tilde{\eta}] = \emptyset$$

$\rightsquigarrow$ simplifies programming

- refinement:

$$[\eta] \subset [\xi],\ \xi_{\nu,k} \in [\eta_\nu] \quad \text{form subsequence of } \eta_\nu$$

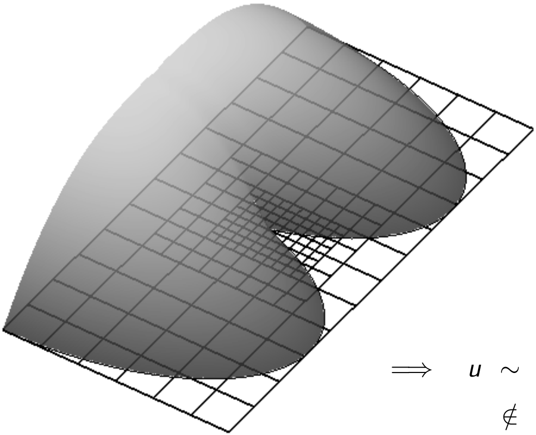
$\rightsquigarrow$ subdivision applicable

- limited overlaps:
  only B-splines of adjacent levels can have common support
  $\rightsquigarrow$ stability

- normalization:
  $\rightsquigarrow$ partition of unity

Approximation of Singularities

$$-\Delta u = 1, \qquad u = 0 \text{ on curved}, \qquad \partial^{\perp} u = 0 \text{ on vertical boundary}$$

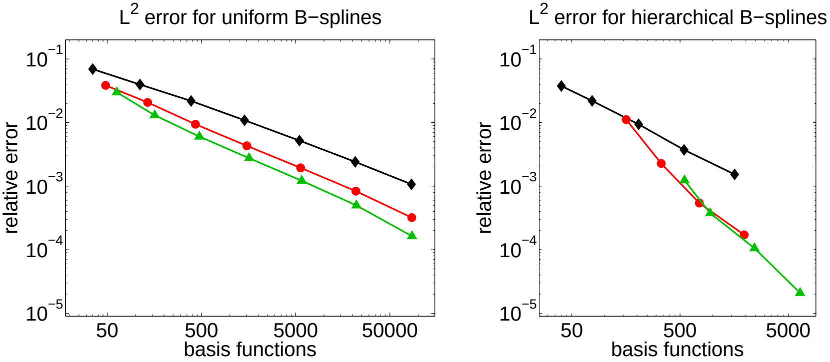


$$\Longrightarrow \quad u \sim r^{0.5528} \quad \text{at reentrant corner},$$
$$\notin H^2$$

Error Behavior

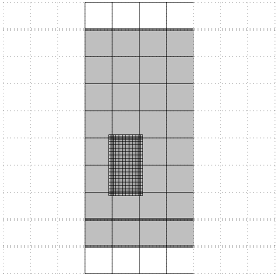
implementation: FEMB package (`http://www.siam.org/books/fr26/`)

limited overlap $\Longrightarrow$ matrix assembly via subdivision

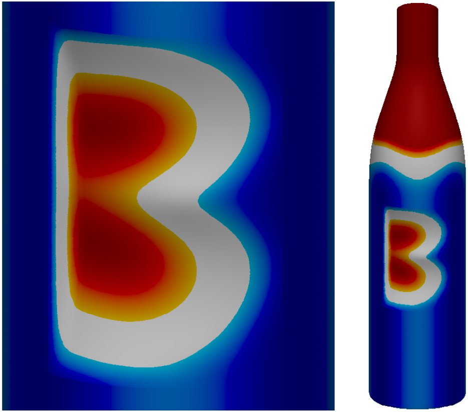


relative errors for B–splines of degree 1 , 2 , 3

Modeling of Surface Details



$\longrightarrow$



hierarchical basis: two levels, $52 + 209$ B-splines