Web-Spline Approximation of Elliptic Boundary Value Problems

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Overview

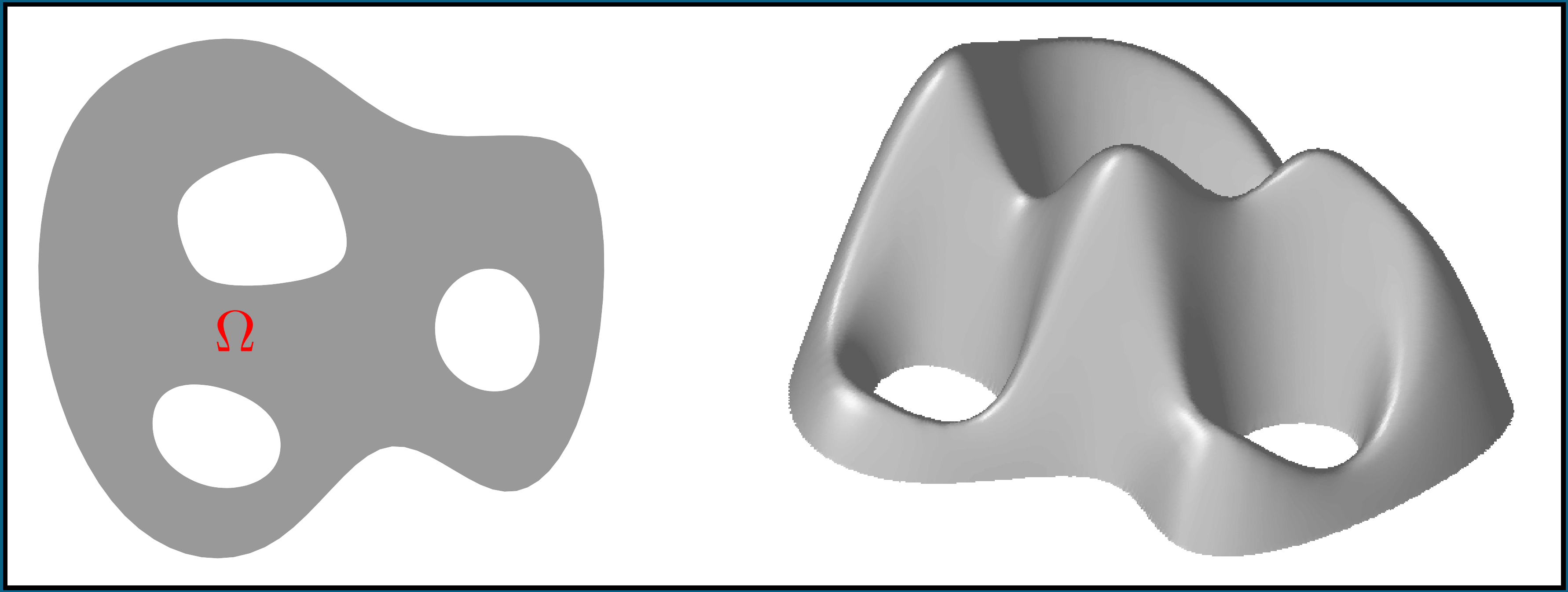
- Model problem
- Standard FE-techniques
- Uniform b-splines
- Weighted extended b-splines
  - Stability
  - Approximation order
- Examples
- Multigrid
- Extensions and further development
- Conclusion

Model problem

On a bounded domain $\Omega \subset \mathbb{R}^m$

we consider Poisson's equation $-\Delta u = f \quad \text{in } \Omega$

with Dirichlet boundary conditions $u = 0 \quad \text{on } \partial\Omega.$



Weak formulation:

$$\int_\Omega \nabla u \nabla \psi = \int_\Omega f\psi, \quad \forall\ \psi \in H_0^1.$$

An approximation in a finite dimensional subspace $\mathbb{B} = \mathrm{span}\{B_i, i \in I\}$

$$\mathbb{B} \ni u_h = \sum_{i \in I} a_i B_i \approx u \in H_0^1$$

is obtained by solving the Galerkin system

$$\sum_{i \in I} \int_\Omega \nabla B_k \nabla B_i \, a_i = \int_\Omega f B_k, \quad k \in I$$

$$\sum_{i \in I} g_{k,i} \, a_i = f_k, \qquad k \in I$$

$$GA = F$$

Objectives:

- fast convergence $u_h \longrightarrow u$ as $h \longrightarrow 0$
- respect boundary conditions
- $\mathrm{cond}\, G_h \sim h^{-2}$
- low dimensional subspace
- efficiency, i.e. number of iterations $\sim 1/h$ or even ~ 1
- practicability

Standard FE-techniques

mesh-based:

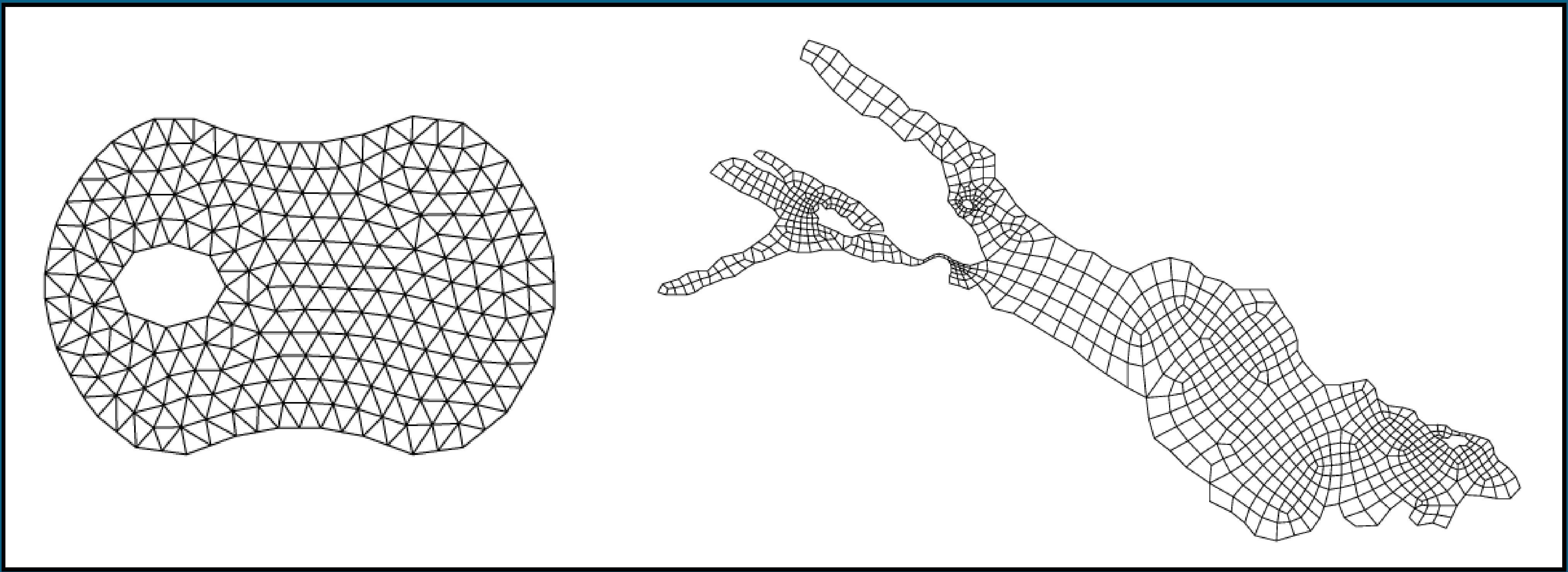
- hat functions
- macro elements (Clough-Tocher, Agyris, Schumaker)

meshless:

- radial basis functions
- wavelets
- hp elements

Hat functions:

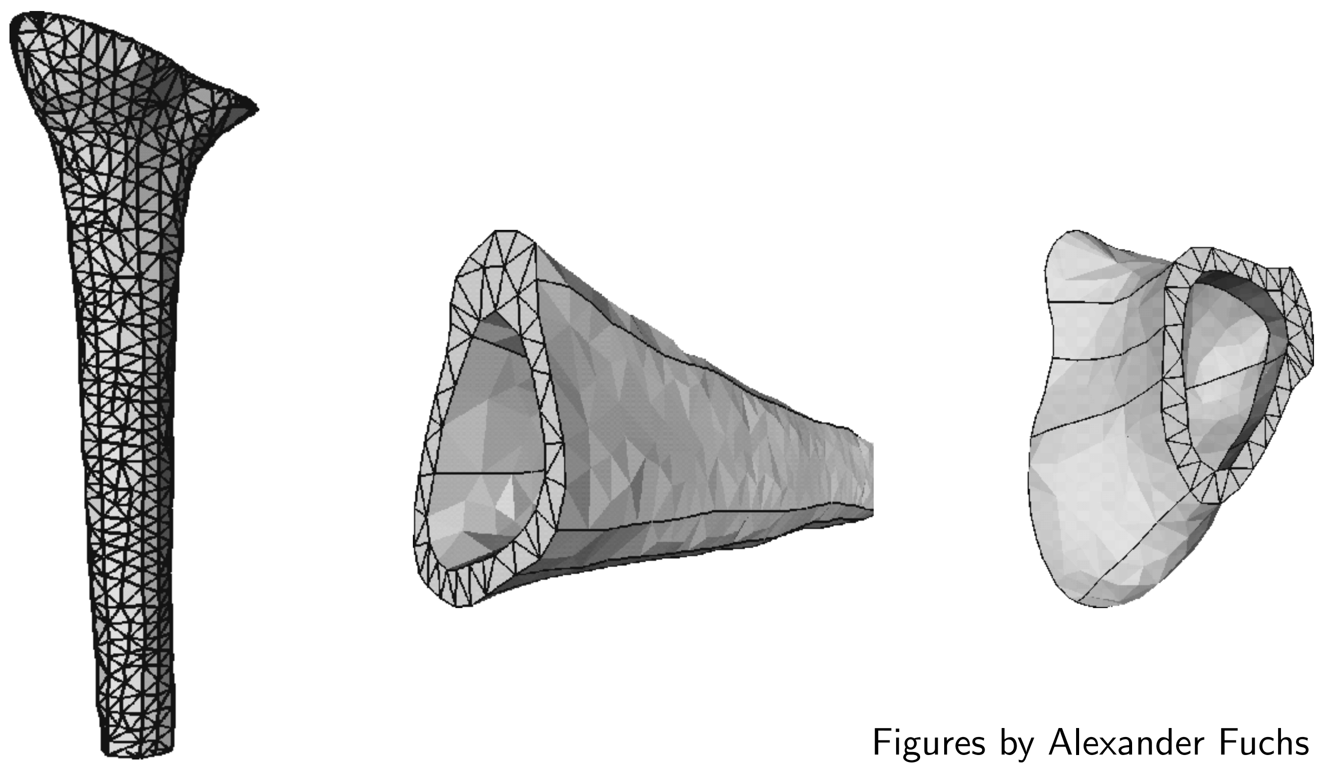
- Based on triangulation (or quadrangulation) of Ω.
- 2d-meshing expensive.



Figures by Dietrich Nowottny

Hat functions:

- Based on triangulation of Ω.
- 2d-meshing expensive.
- 3d-meshing very expensive.



Figures by Alexander Fuchs

Hat functions:

- Based on triangulation of Ω.
- 2d-meshing expensive.
- 3d-meshing very expensive.
- Slow convergence,

$$\|u - u_h\|_0 \sim h^2.$$

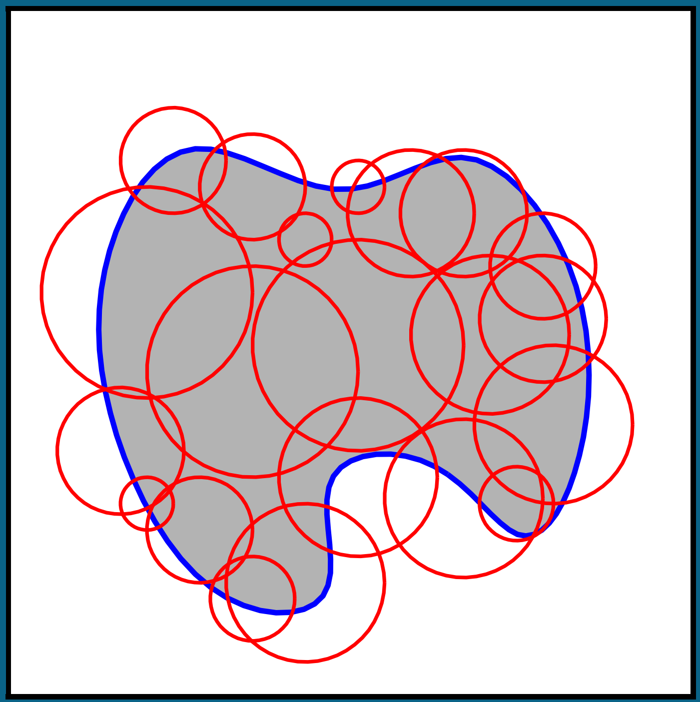
- High dimensional subspaces,

$$\dim \mathbb{B} \sim \|u - u_h\|_0^{-m/2}.$$

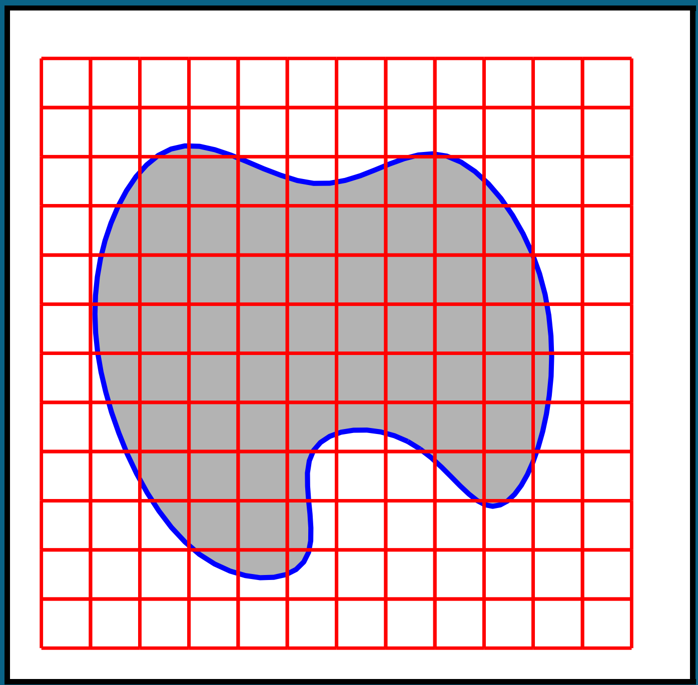
- $\operatorname{cond} G_h \sim h^{-2}$, iff triangulation is uniform.
- Huge amount of code implemented and optimized.

Meshless methods:

unstructured



structured



Main difficulties:

- Obey boundary conditions.

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- Control condition number.

Babuška proposes:

- Lagrange multiplier method
  - saddle point problem
  - indefinite system
  - LBB condition
- Penalty method
  - minimize energy + penalty on boundary deviation
  - balance of terms very delicate

"Both methods have their adherents, . . . , none, however, has gained universal popularity" (Bochev & Gunzberger '98).

Uniform b-splines

The tensor product b-spline basis of order n with knots $h\mathbb{Z}^m$ is

$$\{b_k : k \in \mathbb{Z}^m\}, \quad \operatorname{supp} b_k = h(k + [0,n]^m).$$

Potential benefit:

- No mesh generation required.
- Fast convergence,

$$\|u - u_h\|_0 \sim h^n.$$

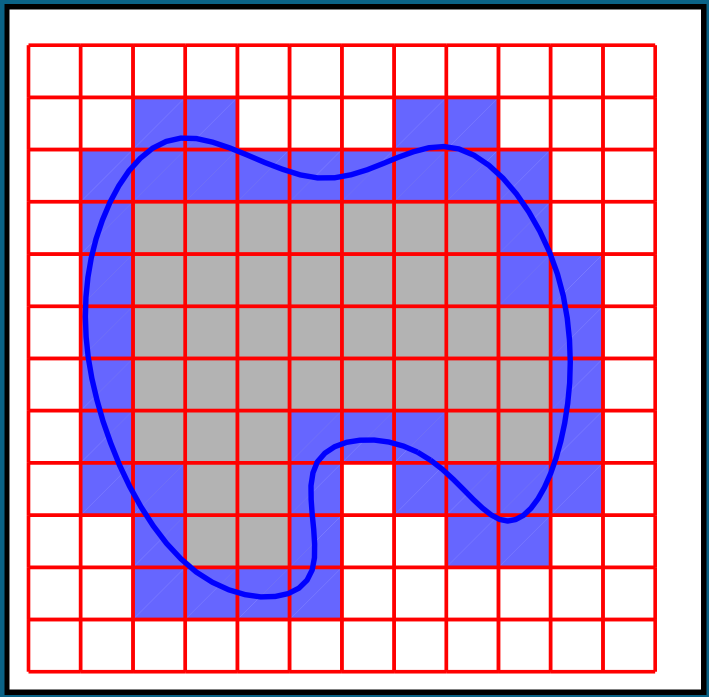
- Low (lowest) dimensional subspace

$$\dim \mathbb{B} \sim \|u - u_h\|_0^{-m/n}.$$

Problems:

❑ Boundary conditions:

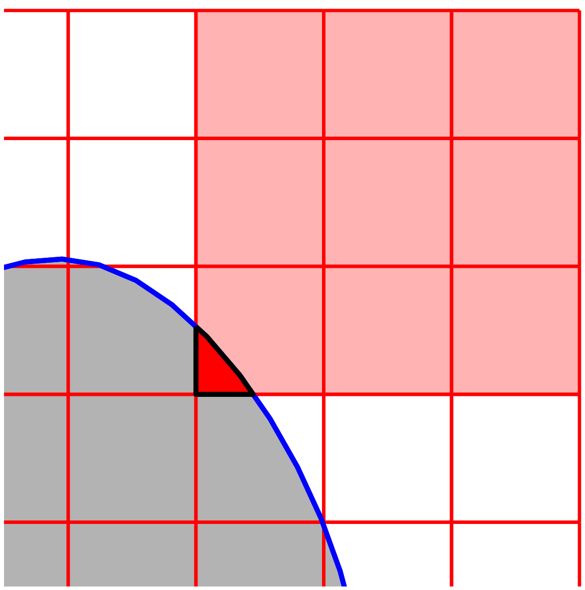
- If a spline is zero on the boundary of Ω, then it vanishes on all intersecting grid cells (in general). This implies a complete loss of approximation power.
- Apply Babuška methods?

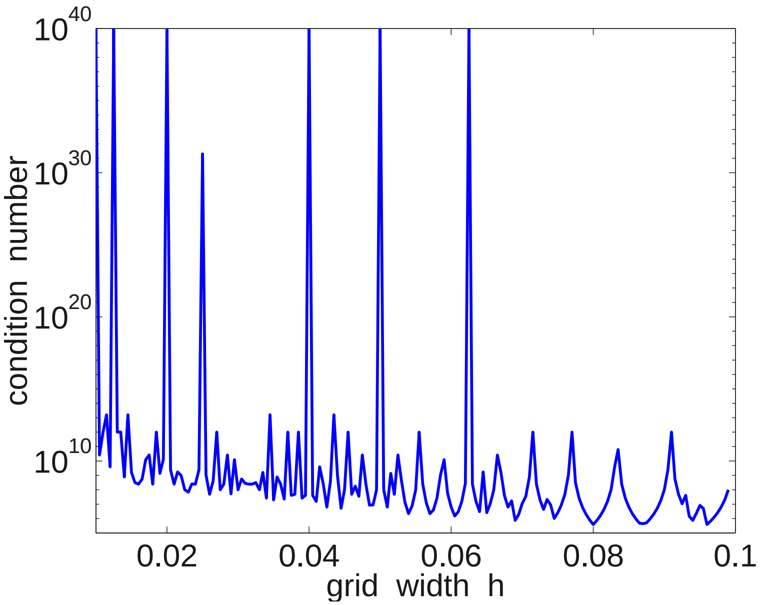


Problems (contd.):

❑ Condition number:

- b-splines with small support in Ω may lead to excessively large condition numbers.
- Leaving out outer b-splines reduces approximation power.
- Just ignore it (brute force)?





Weighted extended b-splines (web-splines)

Partition relevant indices $K := \{k \in \mathbb{Z}^m : \operatorname{supp} b_k \cap \Omega \neq \emptyset\}$:

The inner b-splines with indices

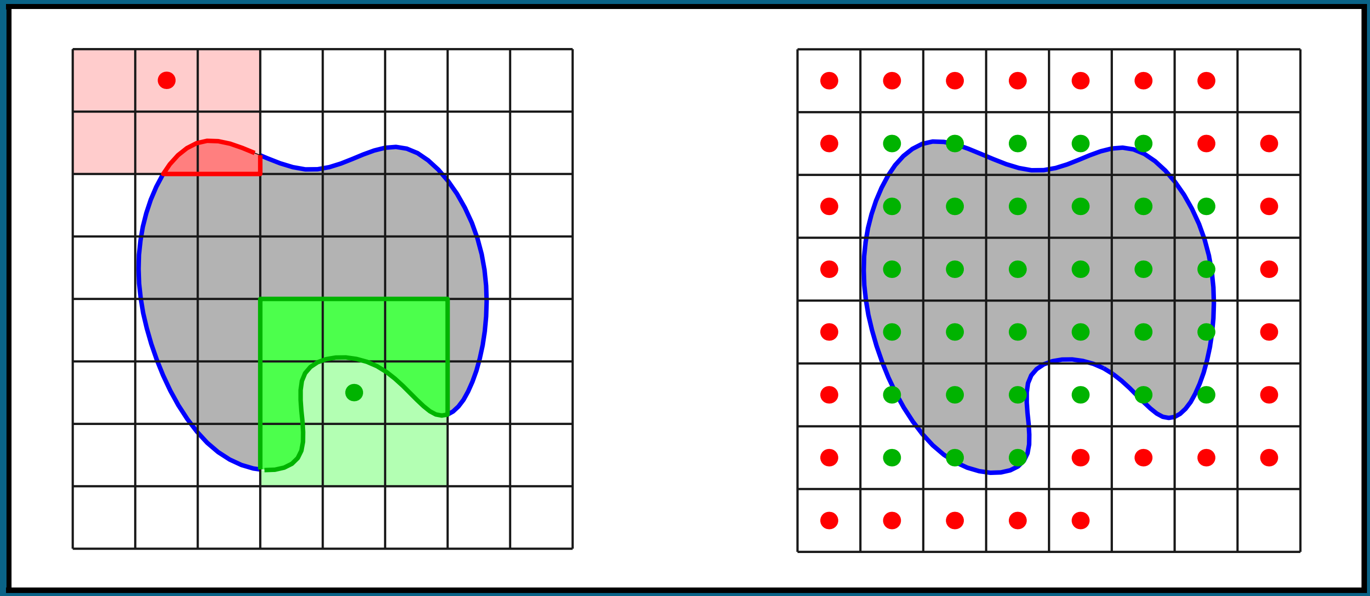
$$I \subset K$$

have at least one grid cell in their support contained in Ω.

The outer b-splines with indices

$$J = K \backslash I$$

have no grid cell in their support contained in Ω .



Extension:

In order to stabilize the basis, the outer b-splines are no longer considered to be independend. Instead, they are coupled with inner b-splines,

$$B_i = b_i + \sum_{j \in J} e_{i,j} b_j, \quad i \in I.$$

- B_i is an extended b-spline, i.e. $\operatorname{supp} B_i \supset \operatorname{supp} b_i$.
- Local extension yields uniformly bounded support,

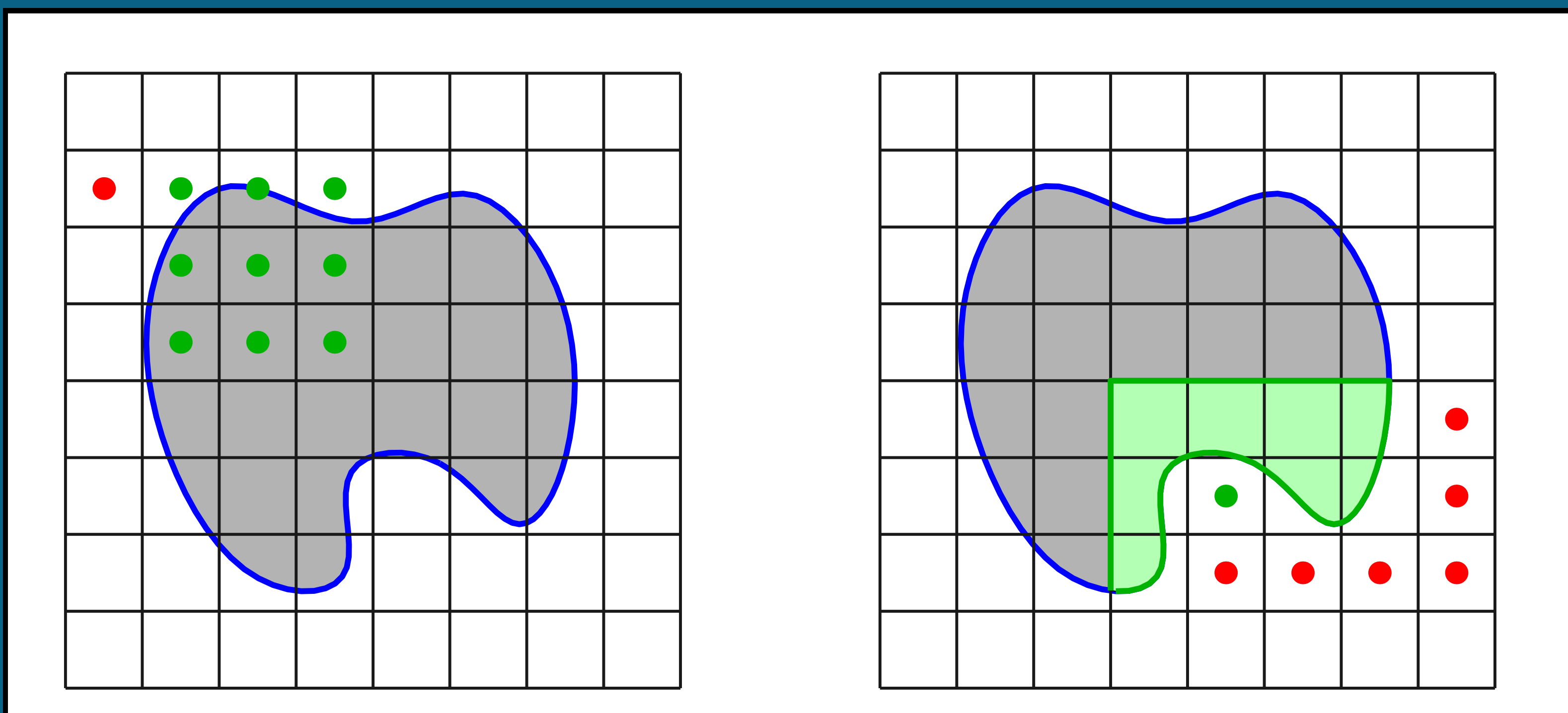
$$e_{i,j} = 0 \quad \text{for } \|i - j\| \succeq 1 \quad \Rightarrow \quad |\operatorname{supp} B_i| \preceq h.$$

  Moreover, most b-splines remain unchanged.
- Choose coefficients $e_{i,j}$ in such a way that all polynomials of order n remain in the span of the extended B-Splines B_i using Marsden's identity,

$$\sum_{k \in K} p(k) b_k \in \mathbb{P}_n(\Omega) \quad \text{iff} \quad p \in \mathbb{P}_n(K).$$

For any outer index $j \in J$ let

- $I(j) \subset I$ be a closest inner array of dimension n^m,
- $J(i) = \{j \in J : i \in I(j)\}$ be the dual index set of $I(j)$.
- $L_i, i \in I(j)$, be the Lagrange polynomials associated with $I(j)$.



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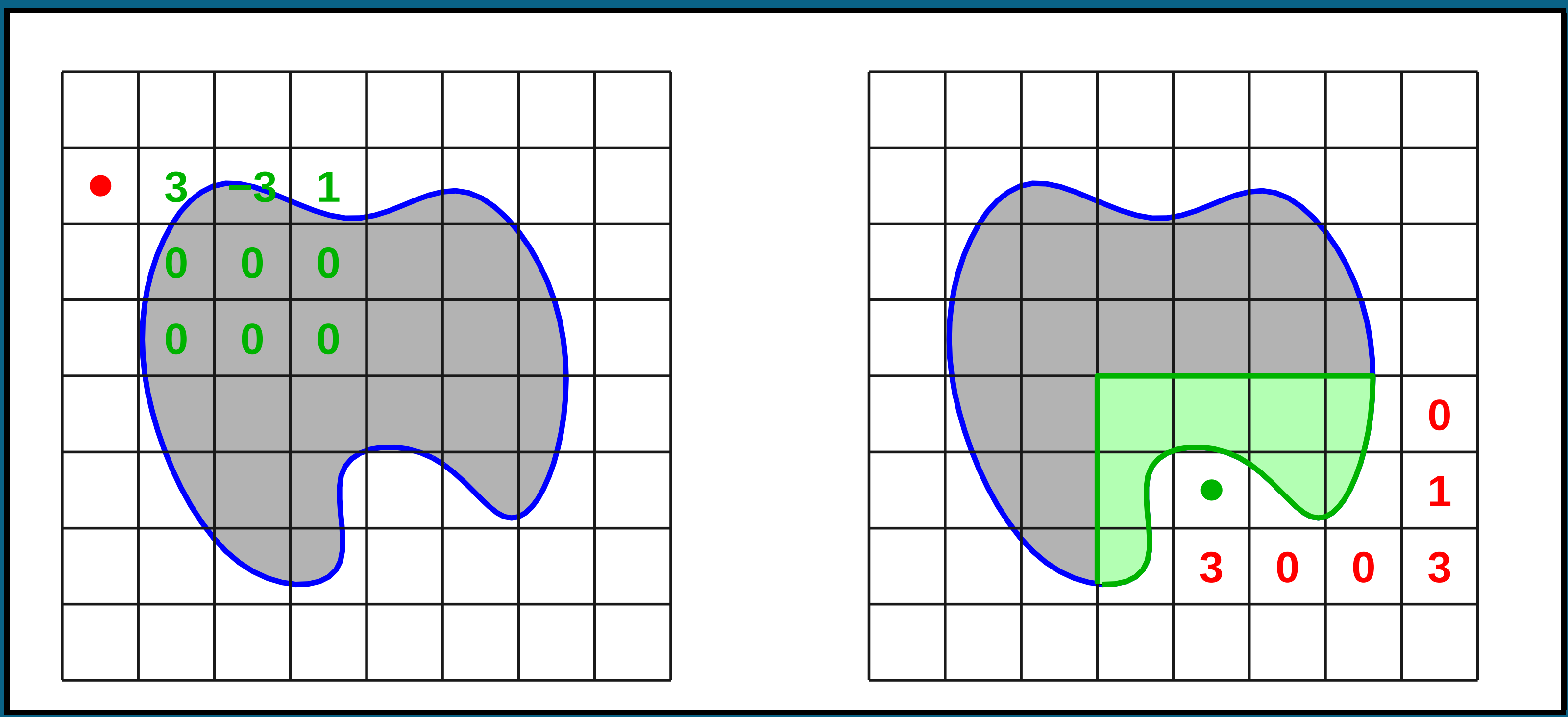
Choosing the coefficients

$$e_{i,j} = \begin{cases} L_i(j) & \text{for } i \in I(j) \\ 0 & \text{else} \end{cases}$$

yields the wanted representation

$$\sum_{i \in I} p(i) B_i = \sum_{k \in K} p(k) b_k .$$



Weighting:

The incorporation of zero boundary conditions is amazingly simple. Let $w : \Omega \to \mathbb{R}_0^+$ be a smooth function equivalent to the boundary distance, i.e.

$$\frac{w(x)}{\mathrm{dist}(x, \partial\Omega)} \preceq 1, \quad \frac{\mathrm{dist}(x, \partial\Omega)}{w(x)} \preceq 1,$$

and in particular

$$w = 0 \quad \text{exactly on } \partial\Omega.$$

Multiplying the extended b-splines B_i by the weight function w yields a basis which satisfies the boundary condition.

Definition:

The web-splines B_i are defined by

$$B_i = \frac{w}{w(x_i)}\Big(b_i + \sum_{j\in J(i)} e_{i,j} b_j\Big), \quad i \in I,$$

where $x(i)$ is the center of a grid cell in $\operatorname{supp} b_i \cap \Omega$.

The web-splines span the web-space

$$\mathbb{B} := \operatorname{span}\{B_i : i \in I\}.$$

Stability

For $\lambda_k, k \in I$, a family of dual functionals for b_i supported on Ω let

$$\Lambda_k = \frac{w(x_k)}{w}\,\lambda_k.$$

Theorem 1: *For $i, k \in I$, the dual functionals Λ_k and the web-splines B_i are uniformly bounded in L_2 with respect to the grid width h, and biorthogonal,*

$$\|B_i\|_0 \preceq 1, \quad \|\Lambda_k\|_0 \preceq 1, \quad \int_\Omega B_i \Lambda_k = \delta_{i,k}.$$

Theorem 2: *The web-basis is stable with respect to the L_2-norm,*

$$\left\| \sum_{i \in I} a_i B_i \right\|_0 \sim \|A\| \ .$$

Theorem 3: *The web-basis satisfies*

$$\Big\|\sum_{i\in I} a_i B_i\Big\|_r \preceq h^{-r}\,\|A\|.$$

Theorem 4: *The spectrum of the Galerkin matrix G_h is bounded by*

$$1 \preceq \varrho(G_h) \preceq h^{-2}.$$

Theorem 5: *The condition number of the Galerkin matrix is bounded by*

$$\operatorname{cond} G_h \preceq h^{-2}.$$

Approximation order

Theorem 6: *Let $u \in H_0^1$ be a smooth function. Then*

$$\|u - v_h\|_r \preceq h^{n-r}, \quad v_h = \mathcal{P}u := \sum_{i \in I} \left(\int u \Lambda_i\right) B_i.$$

Theorem 7: *Let u be a smooth solution of the model problem and $u_h \in \mathbb{B}$ a finite element approximation obtained by solving the Galerkin system. Then*

$$\|u - u_h\|_r \preceq h^{n-r}.$$

Multigrid

The performance of cg-solvers ($\sim h^{-1}$ iterations) can be improved by multigrid methods. These require

- a smoothing operator S, e.g. Richardson's method

$$S : A \to A + \lambda_{\max}^{-1}(F - GA).$$

- a grid transfer operator $\mathcal{P} : \mathbb{B}^{2h} \to \mathbb{B}^{h}$,

$$\mathcal{P} : A^{2h} \to A^{h} = PA^{2h}$$

  with matrix entries

$$p_{\ell,i} = \frac{w(x_\ell^h)}{w(x_i^{2h})} \left(c_{\ell-2i} + \sum_{j \in J^{2h}(i)} e_{i,j}^{2h} c_{\ell-2j} \right).$$

Multigrid Algorithm $U \to W = M(U, F, h)$:

$V = S^\alpha U$	% α smoothing iterations
$\widetilde{F} = P^t(F - GV)$	% residual on coarse grid
if $2h = h_{\max}$	%
$\qquad \widetilde{W} = \widetilde{G}^{-1}\widetilde{F}$	% direct solution on coarsest grid
else	%
$\qquad \widetilde{W} = M^\beta(0, \widetilde{F}, 2h)$	% β multigrid steps
end	%
$W = V + P\widetilde{W}$	% update on fine grid

Theorem 8: *For $\beta = 2$ and α sufficiently large (W-cycle), the multigrid algorithm converges after $O(1)$ iterations. Thus, the complexity for solving the FE-problem reduces to $O(\dim \mathbb{B})$.*

Extensions and further development

- The method potentially applies to many FE problems.
- Hierarchical b-splines can be used for local and adaptive grid refinement.
- The weight function is still subject to optimization.
- Extend the method to non-smooth problems
  - by local refinement,
  - by assymptotic expansion.
- Implementation (3d, multigrid) in progress.

Conclusion

The web-spline method is a promising new FE technique providing the following features:

- Wide range of applicability.
- No mesh generation required.
- High accuracy approximation with relatively few coefficients.
- $O(1)$-convergence with multigrid.
- Based on industrial standard (b-splines).
- Easy to implement (3d integration subtle).