The WEB-Method

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http://www.web-spline.de

History of Finite Elements and Splines

Engineering

Turner, Clough, Martin,
and Topp (1956)
Argyris (1960)
Clough (1960)

de Casteljau (1959),
Bézier (1966)

FEM ← Splines

Rayleigh (1870)
Ritz (1908)
Galerkin (1915)
Courant (1943)
Strang and Fix (1973)

Schoenberg (1946)
de Boor (1972)

Mathematics

Ritz-Galerkin Method

minimize energy subject to boundary conditions

$$Q(u) = \frac{1}{2}a(u,u) - \lambda(u) \to \text{min}, \quad u \in \mathcal{B}$$

finite dimensional approximation

$$u \approx u_h = \sum_{i \in I} c_i B_i$$

with B_i spanning a subspace $\mathcal{B}_h \subset \mathcal{B}$

Ritz-Galerkin system $GC = F$

$$g_{i,i'} = a(B_i, B_{i'}), \quad f_i = \lambda(B_i)$$

Model Problem

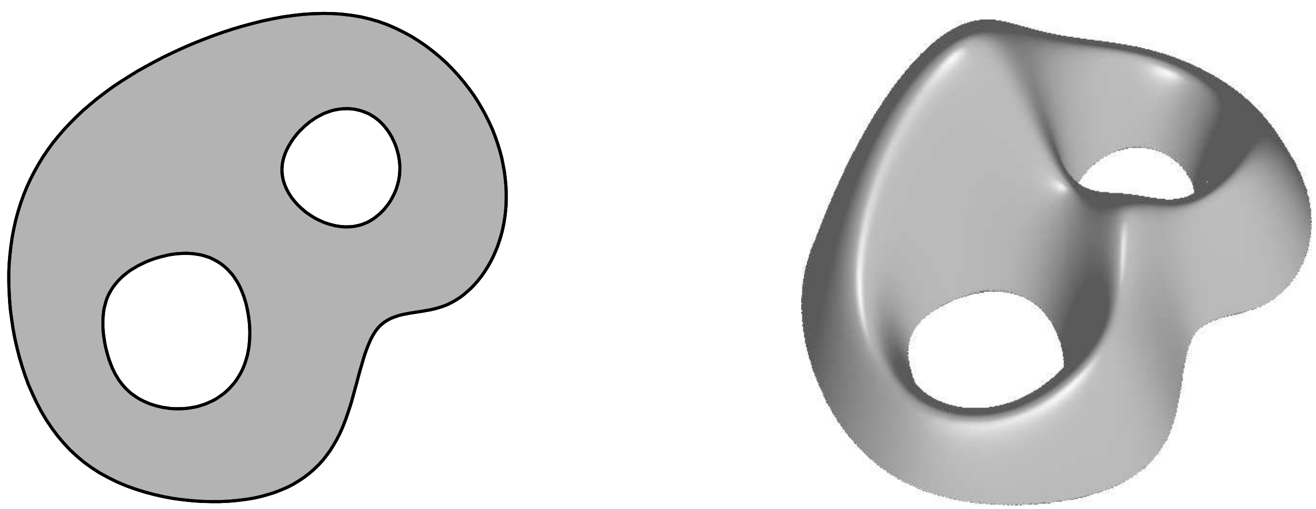
Poisson equation with Dirichlet boundary conditions

$$-\Delta u = f \text{ in } D, \qquad u = 0 \text{ on } \partial D$$

variational formulation

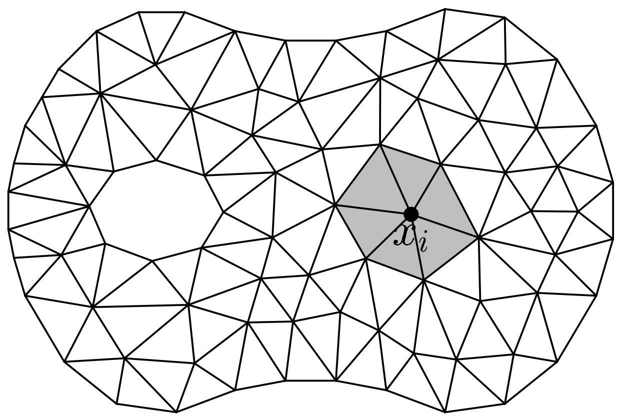
$$a(u,u) = \frac{1}{2}\int_D |\operatorname{grad} u|^2\,, \qquad \lambda(u) = \int_D fu$$

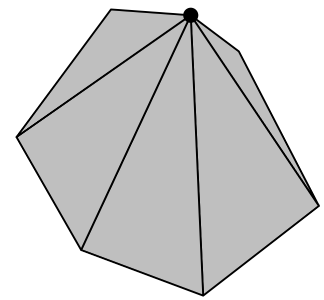
with $u \in \mathcal{B} = \{u:\ \operatorname{grad} u \text{ square integrable}, u_{|\partial D} = 0\}$



Hat-Functions

associated with inner nodes x_i of a triangular mesh

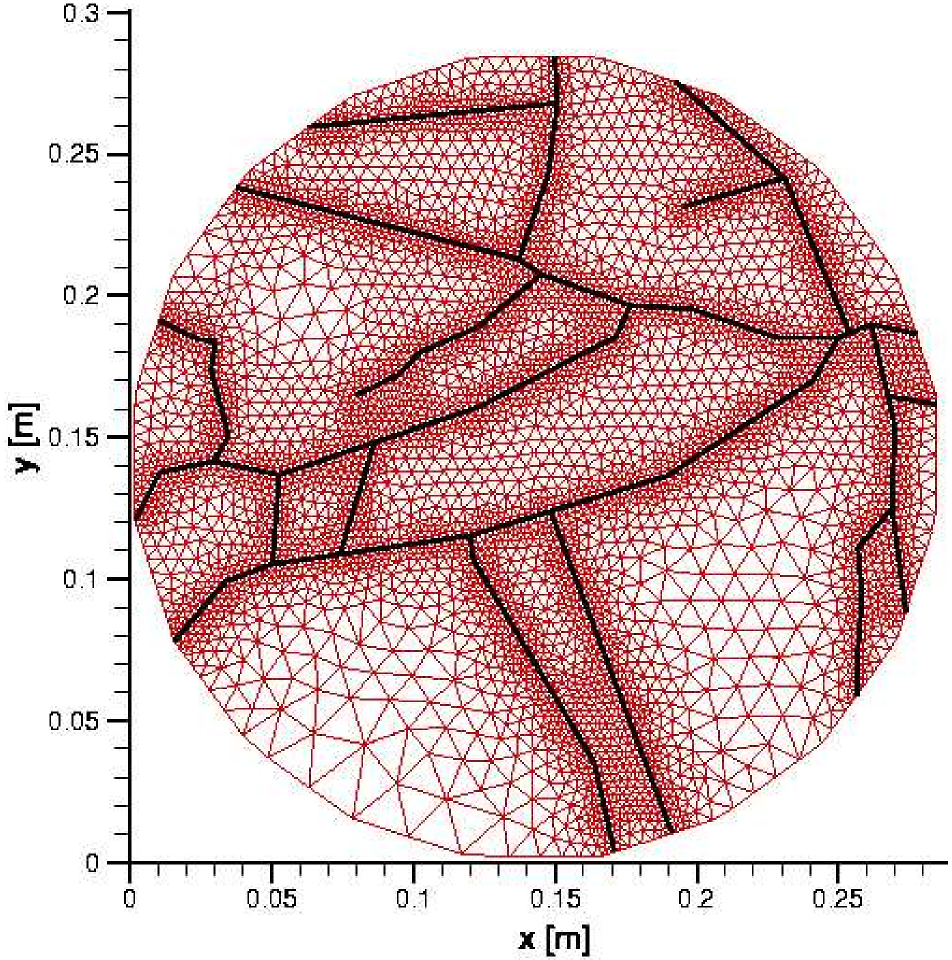




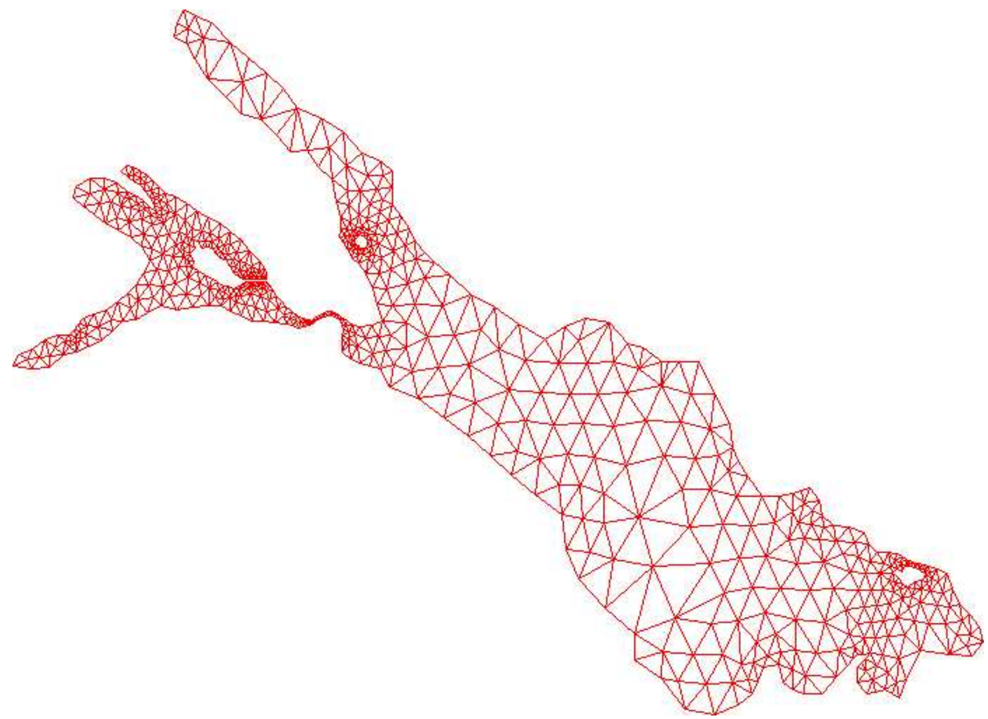
drawbacks:

- time consuming mesh generation
- large number of parameters for high degree elements
- inaccurate boundary approximation
- low smoothness

Almost Regular Planar Triangulations



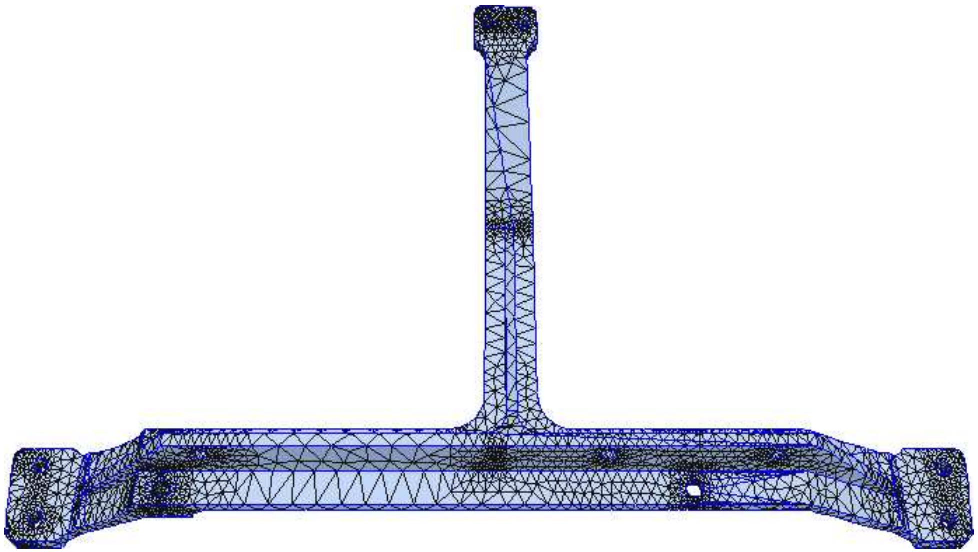
Surface of a core with cracks



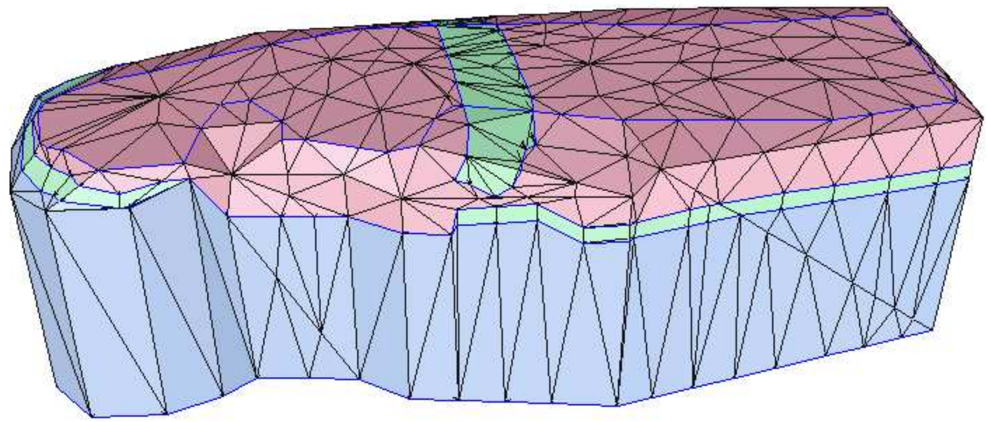
Lake Constance

generated with the ART-Algorithm (courtesy of IVM Fuchs GmbH)

Almost Regular Triangulations of Solids



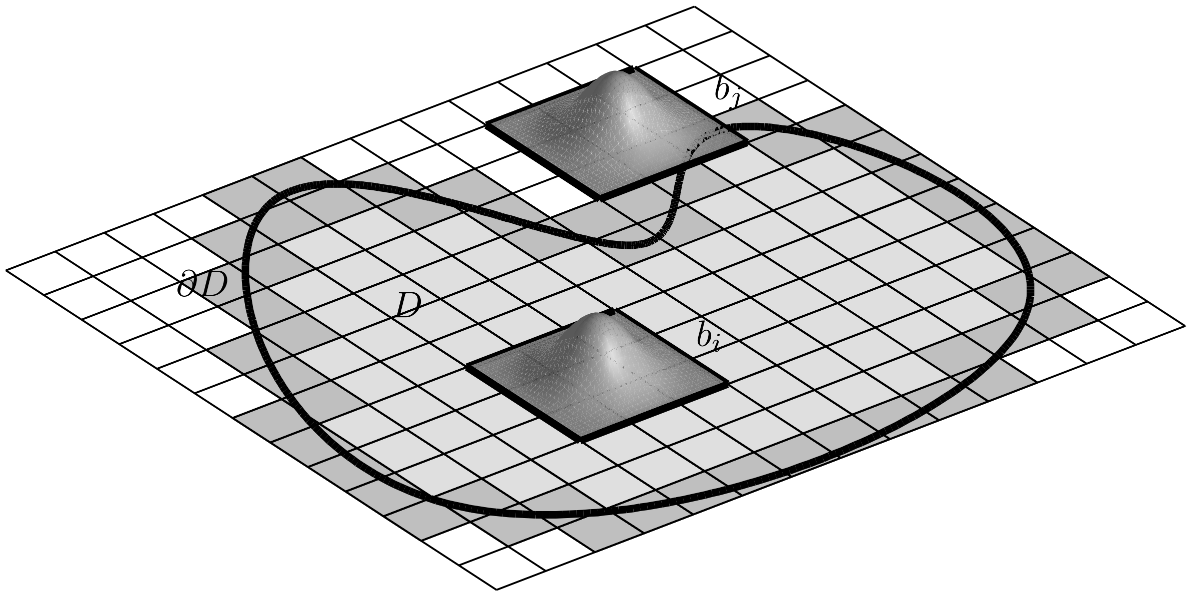
T-Mount



Island Norderney
(two layers meshed with prisms)

generated with the ART-Algorithm (courtesy of IVM Fuchs GmbH)

B-Splines as Finite Elements



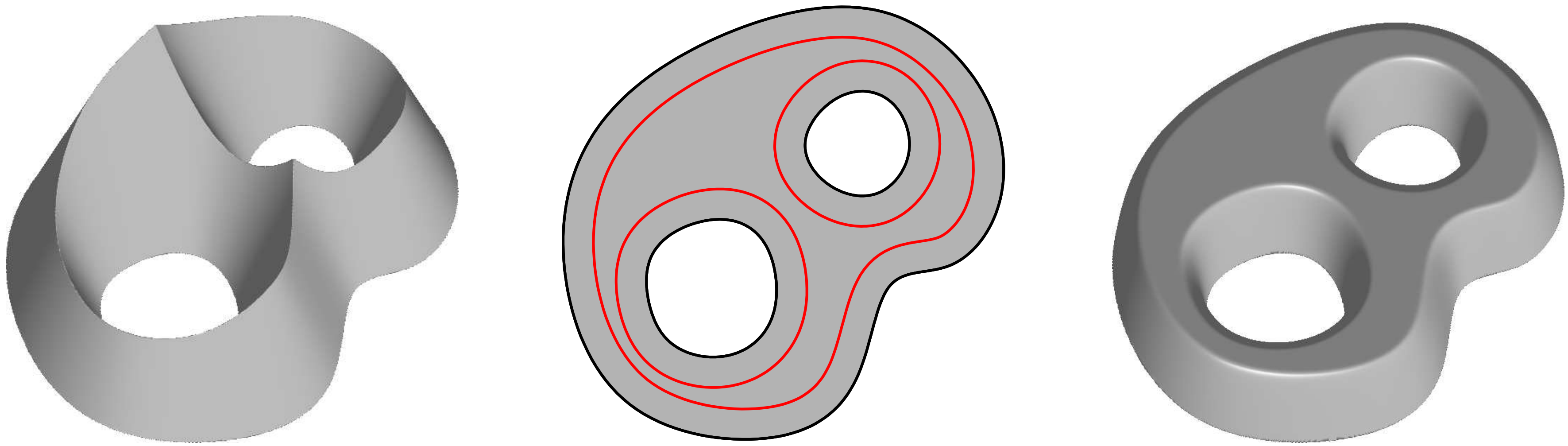
difficulties

- violation of boundary conditions
- instability due to small support

Weight Function

function w, vanishing on $\Gamma \subseteq \partial D$ with order $\gamma > 0$

$$w(x) \asymp \operatorname{dist}(x, \Gamma)^{\gamma}$$

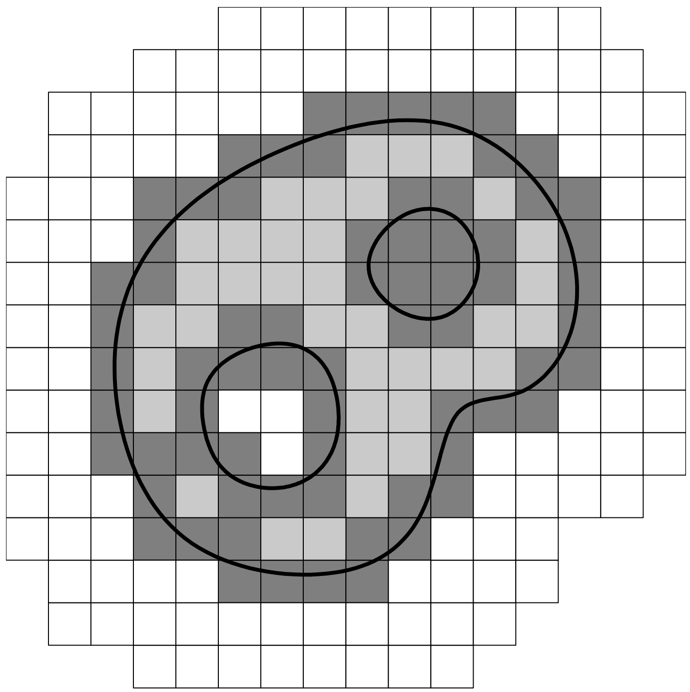
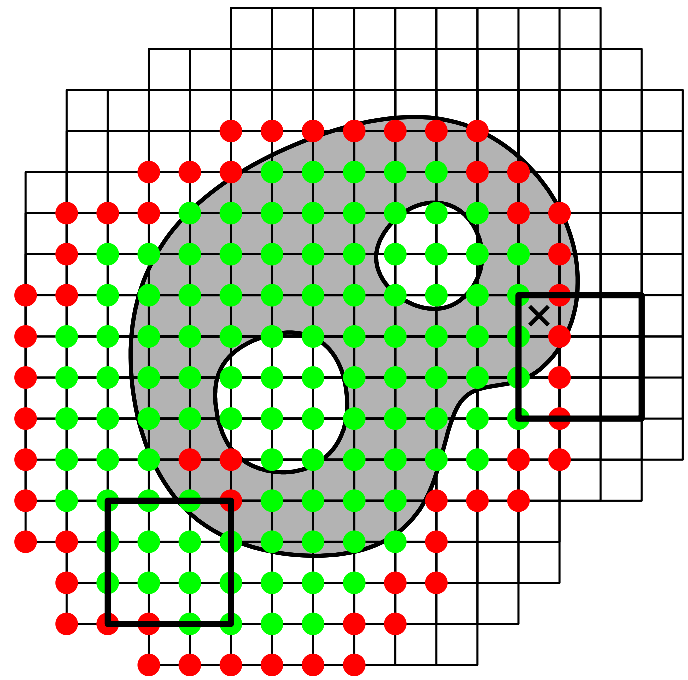


weighted B-splines satisfy boundary conditions

$$wb_k \in \mathcal{B}$$

Classification and Extension

relevant, inner and outer B-splines: $K = I \cup J$

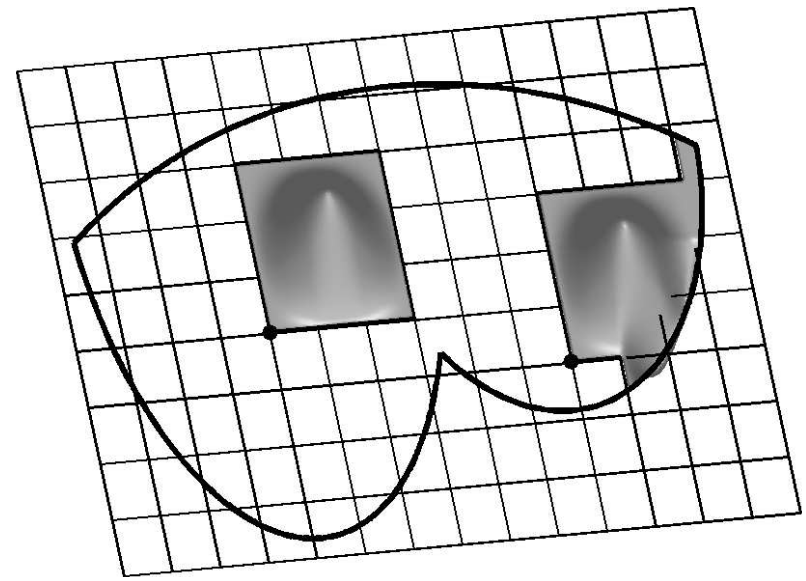
linear combinations maintain stability and accuracy

$$b_i \longrightarrow b_i + \sum_j e_{i,j} b_j$$

Weighted Extended B-Splines

WEB-spline

$$
\begin{aligned}
B_i &= \frac{w}{w(x_i)}\left(b_i + \sum_{j\in J(i)} e_{i,j} b_j\right) \\
&= w \sum_{k\in K} \tilde{e}_{i,k} b_k
\end{aligned}
$$



- weight function vanishes on Γ $\rightarrow$ boundary condition
- support contains grid cell $Q_i \ni x_i$ $\rightarrow$ stability
- linear combinations contain weighted polynomials $\rightarrow$ accuracy

Properties of WEB-Splines

results for WEB-splines with smooth weight function of order 1 on a smooth domain

- stability in $L_2(D)$:

$$h^{m/2}\Big(\sum_i |c_i|^2\Big)^{1/2} \asymp \Big\|\sum_i c_i B_i\Big\|_0,$$

- approximation order for the Sobolev space $H_0^1(D) \cap H^k(D)$:

$$\|u - u_h\|_\ell \preceq h^{k-\ell}\|u\|_k, \quad \ell < k \le n+1$$

- multigrid convergence for the model problem $-\Delta u = f$:

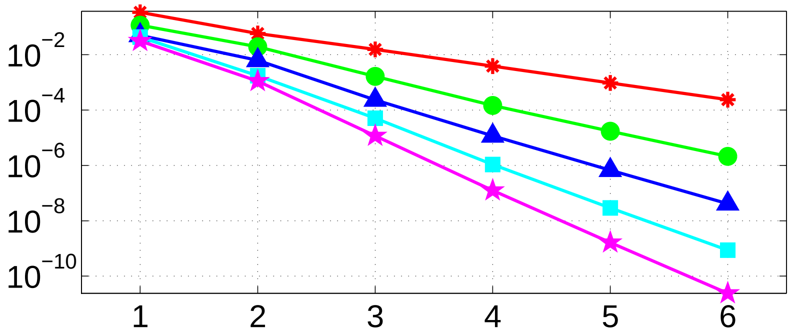
$$\text{solution time} \sim \dim \mathcal{B}_h$$

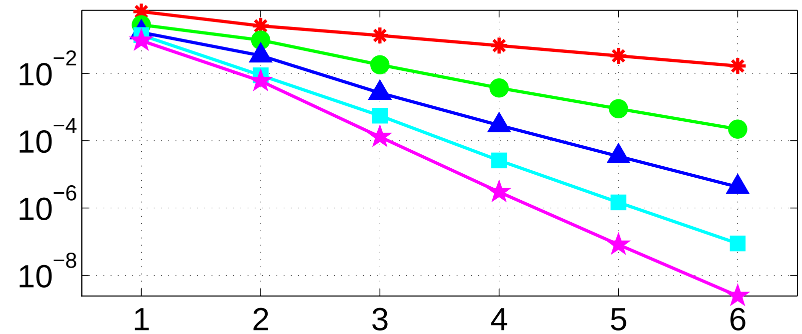
Convergence Rates

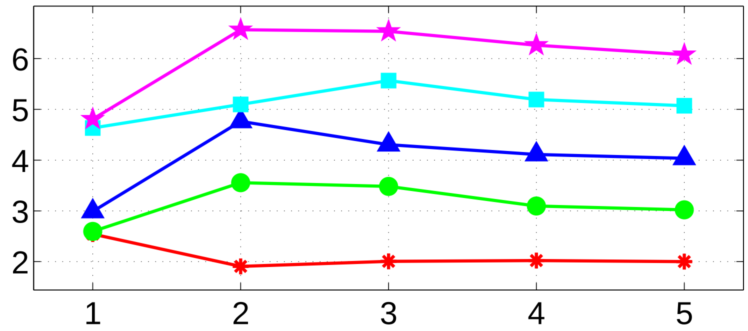
WEB-spline approximation of a model problem with analytic solution

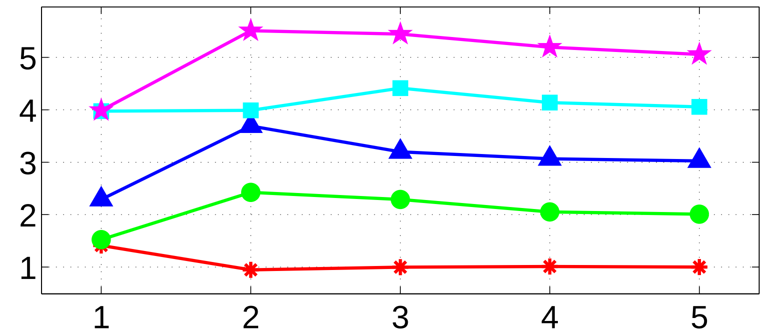
$$-\Delta u = f \text{ in } D, \quad u = 0 \text{ on } \partial D$$

L_2 error and rate (left), H^1 error and rate (right), $h = 2^{-1}, \ldots, 2^{-6}$









Performance of Multigrid

results of the dynamic multigrid solver (dmg) and a standard ssor preconditioned conjugate gradient solver (pcg) to achieve a residual norm $<$ 1e-10

h	dim $\mathcal{B}_h$	pcg iter	pcg rate	dmg iter	dmg rate	pcg/dmg time
2^{-1}	966	19	0.28	60.75	0.67	0.67
2^{-2}	3696	32	0.48	32.31	0.49	1.89
2^{-3}	14383	61	0.68	29.31	0.45	3.72
2^{-4}	56793	120	0.82	24.33	0.38	8.70
2^{-5}	225593	239	0.91	22.33	0.36	19.40
2^{-6}	899281	476	0.95	20.33	0.32	41.76

Linear Elasticity

displacement:

$$(u_1, u_2, u_3) \in \left(H^1_\Gamma\right)^3$$

strain tensor:

$$\varepsilon_{k,l} = \frac{1}{2}\left(\partial_k u_\ell + \partial_\ell u_k\right)$$

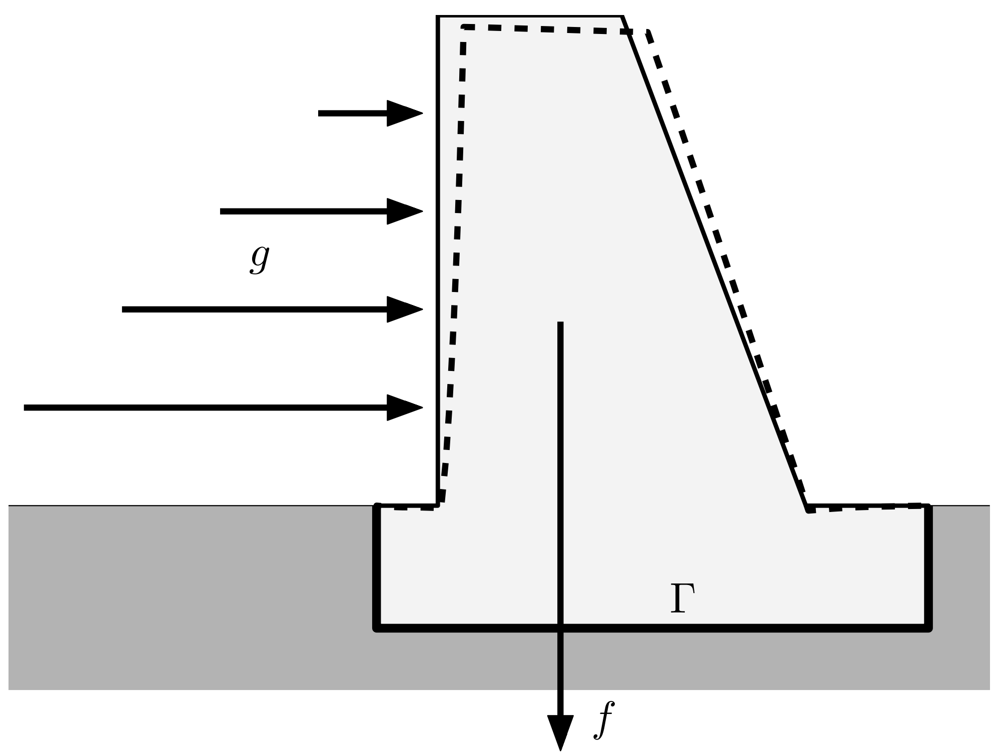
stress tensor:

$$\sigma_{k,l} = \lambda(\operatorname{trace}\varepsilon)\delta_{k,l} + 2\mu\varepsilon_{k,l}$$

variational formulation:

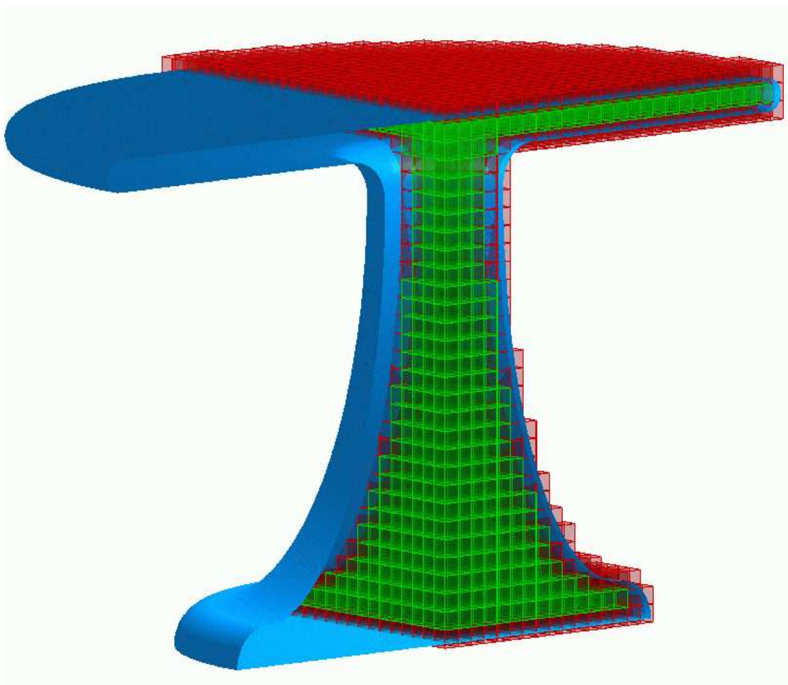
$$a(u,v) = \int_D \sigma : \varepsilon$$

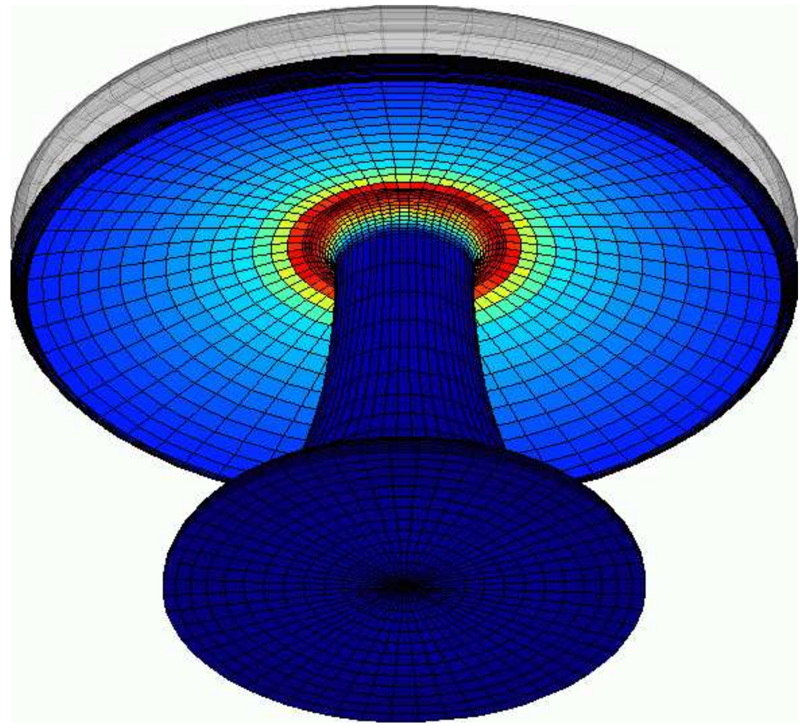
$$\lambda(v) = \int_D f v + \int_{\partial D \setminus \Gamma} g v$$



Stone Table

circular table under gravital force



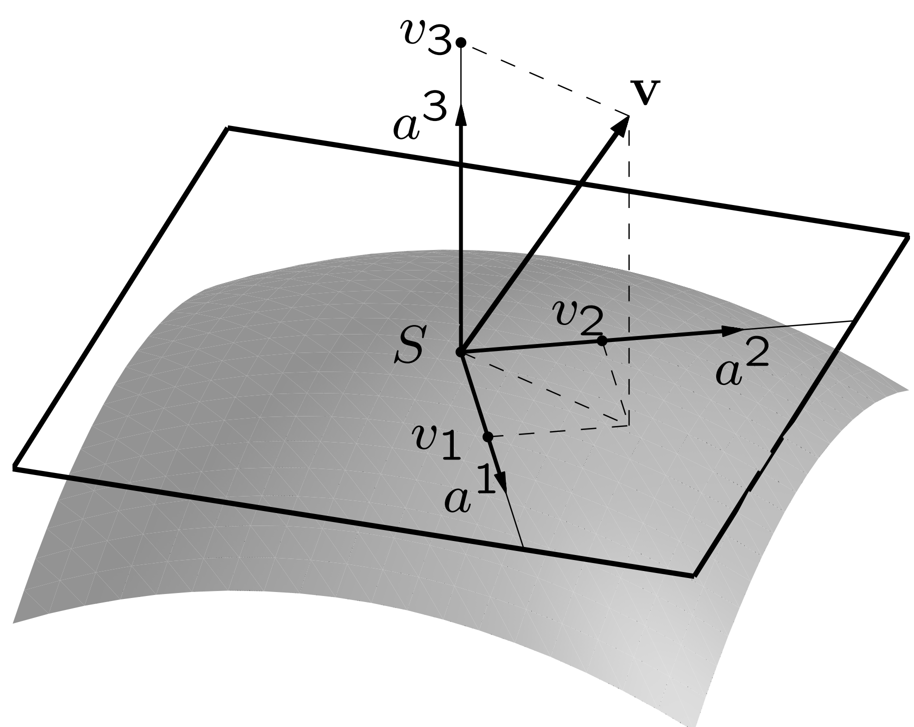


displacement and maximum principle stress were computed using web-splines of degree 2 on a grid consisting of 53x53x30 cubes

Deformation of Thin Shells

variational formulation:

$$f(v) = \int_S p\,v\ , \quad a(u,v) = \int_S e E_S^{ijkl} \left(\gamma_{ij}(u)\gamma_{kl}(v) + \frac{e^2}{12} \varrho_{ij}(u)\varrho_{kl}(v) \right)$$



strain tensor:

$$\gamma_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) + \Gamma_{ij}^k v_k - \beta_{ij} v_3$$

change of curvature tensor:

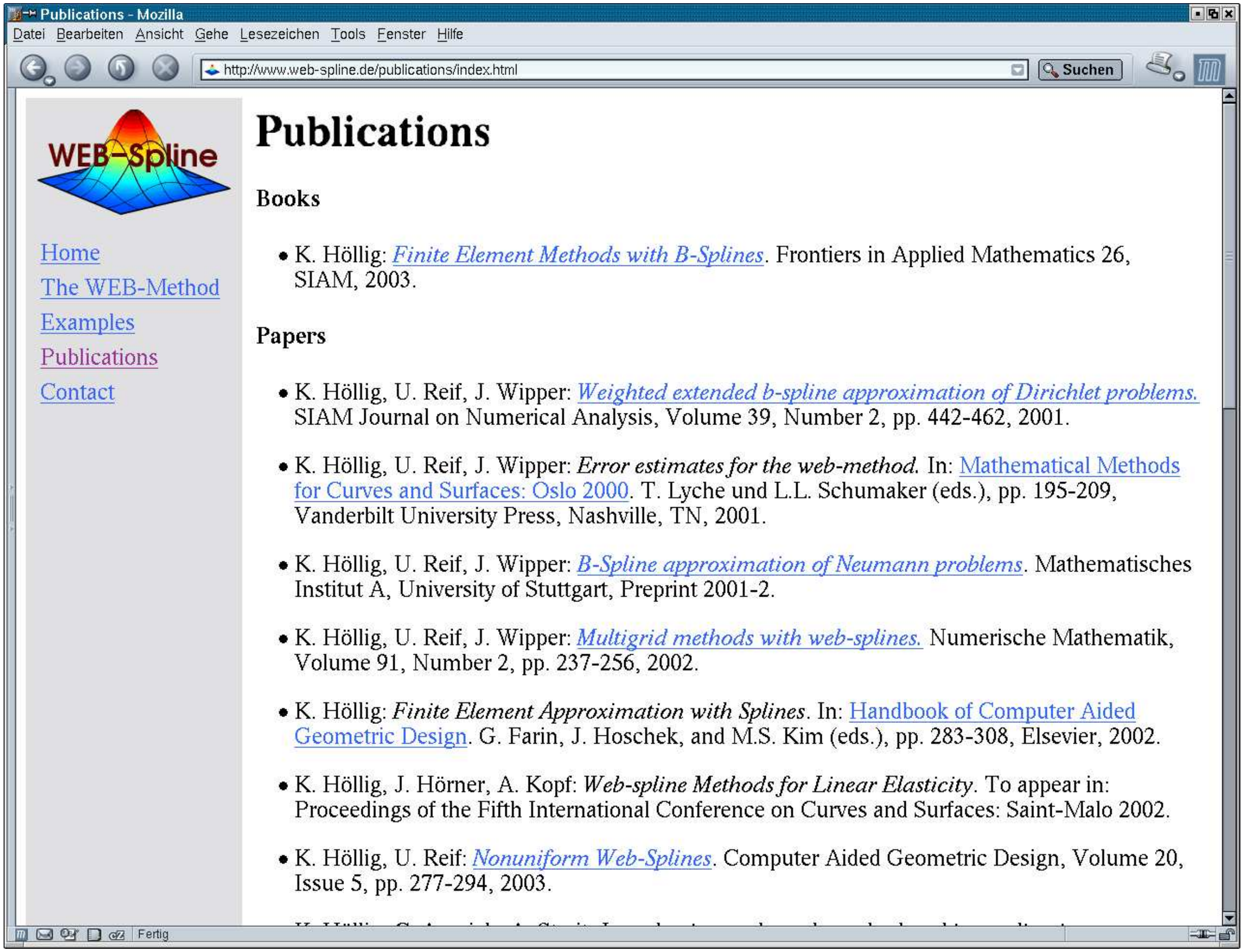
$$\begin{aligned} \varrho_{ij} = v_{3|ij} \quad &- \quad \beta_i^k \beta_{kj} v_3 \quad + \quad \beta_{i|j}^k v_k \\ &+ \quad \beta_i^k v_{k|j} \quad + \quad \beta_j^k v_{k|i} \end{aligned}$$

Summary

WEB-splines combine advantages of B-splines and standard finite elements

- no mesh generation
- high accuracy
- one parameter per grid point
- arbitrary choice of order and smoothness
- adaptive refinement via subdivision
- simple regular data structure
- efficient algorithms
- optimal multigrid solvers

Publications



Publications - Mozilla

Datei Bearbeiten Ansicht Gehe Lesezeichen Tools Fenster Hilfe

http://www.web-spline.de/publications/index.html

WEB-Spline

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Publications

Books

- K. Höllig: *Finite Element Methods with B-Splines*. Frontiers in Applied Mathematics 26, SIAM, 2003.

Papers

- K. Höllig, U. Reif, J. Wipper: *Weighted extended b-spline approximation of Dirichlet problems.* SIAM Journal on Numerical Analysis, Volume 39, Number 2, pp. 442-462, 2001.
- K. Höllig, U. Reif, J. Wipper: *Error estimates for the web-method.* In: Mathematical Methods for Curves and Surfaces: Oslo 2000. T. Lyche und L.L. Schumaker (eds.), pp. 195-209, Vanderbilt University Press, Nashville, TN, 2001.
- K. Höllig, U. Reif, J. Wipper: *B-Spline approximation of Neumann problems*. Mathematisches Institut A, University of Stuttgart, Preprint 2001-2.
- K. Höllig, U. Reif, J. Wipper: *Multigrid methods with web-splines.* Numerische Mathematik, Volume 91, Number 2, pp. 237-256, 2002.
- K. Höllig: *Finite Element Approximation with Splines*. In: Handbook of Computer Aided Geometric Design. G. Farin, J. Hoschek, and M.S. Kim (eds.), pp. 283-308, Elsevier, 2002.
- K. Höllig, J. Hörner, A. Kopf: *Web-spline Methods for Linear Elasticity*. To appear in: Proceedings of the Fifth International Conference on Curves and Surfaces: Saint-Malo 2002.
- K. Höllig, U. Reif: *Nonuniform Web-Splines*. Computer Aided Geometric Design, Volume 20, Issue 5, pp. 277-294, 2003.

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