Finite Element Approximation
with WEB-Splines

Weighted extended b-splines (WEB-splines) provide a new class of finite elements for solving two- and three-dimensional boundary value problems. The resulting WEB-method does not require any grid generation and, as a consequence, can be implemented very efficiently. High accuracy can be obtained with relatively low dimensional approximation spaces. A patent application for this new technology has been filed and (non-) exclusive licence-partners are sought. The participating institutes welcome cooperation for further joint development. A first implementation of the method is available.

Advantages of the WEB-method

- no grid generation
- natural integration in CAD/CAM-systems based on tensor product b-splines
- simple implementation and short computing times
- approximations of arbitrary order of accuracy by appropriate choice of the degree of the trial functions
- low dimensional approximation spaces
- exact fulfilment of boundary conditions
- well suited for multigrid methods and hierarchical refinement

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For further information visit our web-site at www.web-spline.de

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The WEB-Basis

The WEB-method uses tensor product b-splines on regular grids. Hence, in contrast to standard finite element techniques, it is not necessary to construct a mesh for the simulation domain. This eliminates an often rather difficult and time consuming preprocessing step. With b-splines, the approximation order and smoothness can be chosen arbitrarily, so that very accurate solutions are obtained already with comparatively few parameters. Moreover, the WEB-method provides a natural link to the standard geometry description in CAD- and CAM-systems.

Figure 1: Position of the outer (red) and inner (green) b-splines of a quadratic WEB-basis.

Figure 1 illustrates the construction of a quadratic WEB-basis for a simple example. The relevant b-splines are classified into inner b-splines $b_i$ and outer b-splines $b_j$, depending on the portion of their support in the simulation domain. To stabilize the basis, the outer b-splines are then connected to the inner b-splines by forming suitable linear combinations. Finally, possible boundary conditions are incorporated via a weight function.

The resulting basis functions have the form

$$w \left( \frac{w(x_i)}{w(x_i)} \left( b_i - \sum_j e_{i,j} b_j \right) \right).$$

In this expression, the appropriate choice of the coefficients $e_{i,j}$ is crucial. It guarantees that the approximation order and the local support of the b-splines is retained.

Example

For the domain in Figure 1, the Poisson equation $-\text{div}(\text{grad } u) = f$ with homogeneous Dirichlet boundary conditions is solved for

$$u = \sin(\frac{ec}{2}) \quad \text{with}$$

$$e = \frac{x^2}{16} + \frac{y^2}{9} - 1 \quad \text{and}$$

$$c = 1 - (x + 3/4)^2 - (y - 1/2)^2$$

(cf. Figure 2).

Figure 2: Solution $u$ of the Poisson equation.

Figure 3 shows the relative error in the $L_2$-norm versus the number of basis functions for the WEB-bases with degrees 1 to 5. The results are compared with a standard finite element method using hat functions on a triangulation (diamond-marker).

Figure 3: Comparison of the WEB-method with a standard finite element method.