B-SPLINE APPROXIMATION OF NEUMANN PROBLEMS

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Abstract. We describe a new finite element method for Neumann problems using cardinal splines. The essential idea is the construction of a stable basis, so that B-splines with very small support in the domain do not lead to an excessively large condition number of the Galerkin matrix. This simple approach does not require any grid generation and yields smooth, high order accurate approximations with relatively low dimensional subspaces.

1. Introduction

One of the advantages of finite elements is the flexibility in the choice of the approximation methods. If we denote by B_k the basis functions of a finite element subspace $\mathbb{B}_h$ with mesh width h, the following, fairly general conditions are sufficient to guarantee stability and convergence for solving standard second order elliptic problems (cf., e.g., [9]).

Local Support: $|\operatorname{supp} B_k| \preceq h.$ $(\mathbf{B_L})$

Normalization: $\|B_k\|_0 \preceq 1,\ \|B_k\|_1 \preceq h^{-1}.$ $(\mathbf{B_N})$

Stability: $|a_i| \preceq \|\sum_k a_k B_k\|_{0,\operatorname{supp} B_i}.$ $(\mathbf{B_S})$

Accuracy: $\mathbb{B}_h$ contains polynomials of order n. $(\mathbf{B_A})$

Here, $|Q|$ denotes the diameter of a set Q and

$$\|v\|_\ell = \|v\|_{\ell,\Omega} = \left(\sum_{|\alpha|\le\ell} \int_\Omega |D^\alpha v|^2 \right)^{1/2}$$

the Sobolev norm of a function v on a domain $\Omega \subset \mathbb{R}^m$. Moreover, we write

$$a \preceq b,$$

if $a \le cb$ with a constant c, which does not depend on the mesh width h. Finally, we denote later on by $\|A\|$ the 2-norm of a vector A.

Of course, finite element basis functions have to conform to the variational formulation. This usually requires that certain boundary conditions are satisfied. An exception are problems with natural boundary

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conditions, which do not have to be incorporated into the variation functional. For such problems, we can use finite elements on regular partitions, avoiding the time consuming grid generation process. For example, a natural choice are translates of tensor product B-splines of coordinate order n

$$b_k(x) = h^{-m/2} b(x/h - k), \quad k \in \mathbb{Z}^m,$$

which are $(n-2)$-times continuously differentiable and have coordinate degree $n-1$ in each mesh cell $Q_\ell = \ell h + [0,h]^m$. Then,

$$\operatorname{span}\{b_{k|\Omega} : \ \Omega \cap \operatorname{supp} b_k \neq \emptyset\}$$

is a possible finite element subspace for a problem on a bounded domain $\Omega \subset \mathbb{R}^m$.

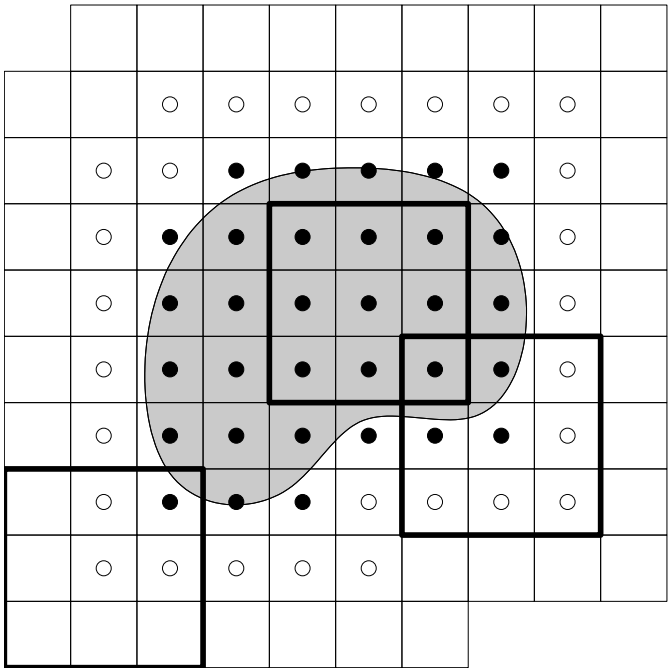

FIGURE 1. Support of quadratic B-splines and arrangement of outer (○) and inner (●) B-splines.

Figure 1 gives an example of a finite element subspace of order 3. We have marked the centers of the supports of all B-splines used in the basis. Moreover, we highlighted the support of three basis functions showing the influence of the boundary.

A disadvantage is the lack of uniform stability of the basis, which is caused by the B-splines

$$b_j, \quad j \in J$$

(marked with white discs in Figure 1), which have only very small support in Ω. While $(\mathbf{B_L})$, $(\mathbf{B_N})$, and $(\mathbf{B_A})$ are valid, the stability

condition ($\mathbf{B_S}$) does not hold. Nevertheless, numerical experiments in [7] yield rather good results. However, the instability of the basis leads to an excessively large condition number of the Galerkin matrix and probably is the reason, why this simple approach has not been pursued.

We show in this paper how to stabilize the B-spline basis. The idea is to adjoin the B-splines b_j, $j \in J$, to the B-splines

$$b_i, \quad i \in I$$

(marked with solid discs in Figure 1), which have at lest a complete mesh cell Q_ℓ in their support. We form linear combinations

$$B_i := b_i + \sum_{j \in J} e_{i,j} b_j \tag{1}$$

with uniformly bounded coefficients $e_{i,j}$, which are nonzero only if $\|i - j\|$ is small, i.e., if ih is close to the boundary and jh are neighboring lattice points (cf. Figure 3). The crux of the construction is that condition ($\mathbf{B_A}$), which guarantees full approximation power of the spline space, remains valid. Thus, the resulting spline space meets all requirements of standard finite element approximations.

To focus on the essential features of our approach, we consider the Neumann problem for Poisson's equation as a model problem. A few basic facts about this example are reviewed in section 2 (cf. e.g. [2]). In section 3, we describe the new finite element subspace and show that it meets conditions ($\mathbf{B}$). The performance of our method is illustrated in section 4. Finally, section 5 discusses possible extensions of our approach, which in principle applies to any elliptic problem with natural boundary conditions.

2. Model Problem

To describe the approximation by B-splines in a simple setting, we consider the Neumann problem

$$\begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega \\ \partial^\perp u & = & 0 & \text{on } \partial\Omega \end{array}, \tag{2}$$

where $\partial^\perp$ denotes the normal derivative at the boundary. We assume that the data f as well as the boundary $\partial\Omega$ are smooth, and that the compatibility condition

$$\int_\Omega f = 0$$

is satisfied. Then, there exists a smooth solution, which is unique up to a constant. We choose this constant, so that the integral of u over Ω is zero, too. In effect, we obtain a unique solution in

$$H_1^\perp := \left\{ v : \ \|v\|_1 < \infty, \int_\Omega v = 0 \right\},$$

a space, which is also convenient for the variational formulation. This asserts that the solution u minimizes the functional

$$\varphi(v) = \frac{1}{2}\int_\Omega \nabla v(\nabla v)^{\mathrm{t}} - \int_\Omega fv$$

among all $v \in H_1^\perp$.

The finite element approximation

$$u_h = \sum_i a_i B_i$$

minimizes φ over $\mathbb{B}_h^\perp := \mathbb{B}_h \cap H_1^\perp$. Expressing $\varphi(u_h)$ in terms of the coefficients A yields the quadratic form

$$\begin{aligned}\varphi_h(A) &= \frac{1}{2}\int_\Omega \sum_{i,k} a_i \nabla B_i (\nabla B_k)^{\mathrm{t}} a_k - \int_\Omega f \sum_i a_i B_i \\ &=: \frac{1}{2} A^{\mathrm{t}} G_h A - F^{\mathrm{t}} A.\end{aligned}$$

We note, that the Galerkin matrix G_h is merely positive semi-definite, since

$$G_h A^\perp = \mathbf{0},$$

if $a_i^\perp$ are the coefficients in the representation of the constant function $v_h = 1$. However, this lack of uniqueness does not cause any problems for standard interative methods. For example, the iterates of the conjugate gradient method will all remain in a hyperplane with normal $A^\perp/\|A^\perp\|$, simply because the gradient of φ_h is orthogonal to $A^\perp$ for all arguments. In general, the limit will not have integral zero over Ω. But, this can be achieved by adding a multiple of $A^\perp$ to the coefficients, which yields the proper approximation $u_h \in \mathbb{B}_h \cap H_1^\perp$.

We now describe two basic theorems, which follow from conditions (**B**) and hence will also hold for the new B-spline approximations. They serve as examples for other typical results about finite elements. Everything is well known, so we will be **very** brief.

By a variant of Céa's Lemma,

$$\|u - u_h\|_1 \preceq \inf_{v_h \in \mathbb{B}_h \cap W_1^\perp} \|u - v_h\|_1,$$

recalling that we are assuming $\int_\Omega u = \int_\Omega u_h = 0$. Hence, deriving an error estimate in the energy norm is reduced to a pure approximation problem.

Theorem 1. $\|u - u_h\|_1 \preceq h^{n-1}\|u\|_n$.

An elegant proof uses a quasi-interpolant

$$Pu := \sum_i \left(\int \lambda_i u\right) B_i$$

as approximation operator. The coefficients of this linear projection onto $\mathbb{B}_h$ are computed using dual functions λ_i, which have no larger support than B_i and satisfy

$$\|\lambda_i\|_0 \preceq 1, \quad \int \lambda_i B_k = \delta_{i,k}.$$

They represent the linear functionals

$$\lambda_i : \sum_k a_k B_k \mapsto a_i,$$

which, by condition $(\mathbf{B_S})$ are uniformly bounded. Since, by $(\mathbf{B_A})$, polynomials of order n are contained in $\mathbb{B}_h$ and reproduced by P, standard arguments, using $(\mathbf{B_L})$ and $(\mathbf{B_N})$, yield the optimal approximation order.

The average convergence rate of the conjugate gradient iteration for minimizing the discrete quadratic form φ_h satisfies

$$\varrho_{\text{cg}} \leq \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}, \quad \kappa = \text{cond}_2 G_h = \frac{\lambda_{\max}}{\lambda_{\min}},$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the extreme positive eigenvalues of the Galerkin matrix G_h (cf., e.g., [5]). For standard finite elements $\kappa \preceq h^{-2}$, which yields the following bound.

Theorem 2. $1 - \varrho_{cg} \succeq h$.

The two extreme eigenvalues can be computed by maximizing and minimizing the Rayleigh quotient

$$r(A) = \frac{A^t G_h A}{A^t A} = \frac{\int \|\nabla v_h\|^2}{\|A\|^2}, \quad v_h = \sum_i a_i B_i,$$

over all A orthogonal to the vector $A^\perp$ in the kernel of G_h. From conditions $(\mathbf{B_L})$ and $(\mathbf{B_N})$ we can deduce that

$$\|v_h\|_1^2 \preceq h^{-2}\|A\|^2,$$

which implies $\lambda_{\max} = \max_{A \perp A^\perp} r(A) \leq h^{-2}$. The lower bound for $\lambda_{\min} = \min_A r(A)$ is more subtle. It uses the stability condition $(\mathbf{B_S})$, which implies

$$\|A\| \preceq \|v_h\|_0.$$

Together with

$$\|v_h\|_0^2 \preceq \int \|\nabla v_h\|^2$$

this yields $\lambda_{\min} \succeq 1$ and thus establishes the estimate for κ and ϱ_{cg}.

The last inequality follows from the Bramble Hilbert Lemma [3], if there exists a bounded functional λ, vanishing on v_h. A possible choice is

$$\lambda v = \sum_i \left(\int \lambda_i v \right) a_i^\perp / \|A^\perp\|.$$

We have $\lambda(v_h) = 0$, since the coefficients of v_h are orthogonal to $A^\perp$ and

$$|\lambda v|^2 \preceq \sum_i \left| \int \lambda_i v \right|^2 \preceq \sum_i \|v\|^2_{0,\operatorname{supp} B_i} \preceq \|v\|^2_0,$$

because of the local support of the dual functions.

3. B-spline basis

In this section we construct a subspace $\mathbb{B}_h$ of cardinal splines on Ω, which satisfies conditions (**B**). As we already mentioned in the introduction, the idea is to stabilize the B-spline basis while retaining full approximation power. Before describing this in detail, it is instructive to consider a univariate example. The top of Figure 2 shows the B-spline basis of piecewise linear functions on the interval $\Omega = [-\varepsilon, 1]$[1]. There is one B-spline $b_j(x) = h^{-1/2} b(x/h - j)$, $j = -1$ with small support in Ω. A simple computation shows that we loose the full approximation order, if we omit b_j. The reason is that constants and linear functions can no longer be represented. To see, how we can retain linear precision, we consider the identity

$$\alpha + \beta x =: p(x) = \sum_{k \geq 1} \underbrace{p(kh + h) h^{1/2}}_{a_k} b_k(x), \quad x \in \Omega. \tag{3}$$

As is clear from the Figure, $h^{-1/2} a_k$ is the value of p at the maximum of b_k. The scaling factor is due to the normalization ($\mathbf{B_N}$). Since a_k is a linear function of k, we can compute a_{-1} from a_0 and a_1 by interpolation,

$$a_{-1} = 2a_0 - a_1.$$

Rewriting (3) as

$$p = a_0(b_0 - 2b_{-1}) + a_1(b_1 - b_{-1}) + \sum_{k>1} a_k b_k.$$

We see, that it is possible to represent linear functions with the modified basis functions

$$B_0 := b_0 - 2b_{-1}, \; B_1 := b_1 - b_{-1}, \quad B_k := b_k, \; k > 1.$$

The new basis, shown at the bottom of Figure 2, is obviously stable, since $B_i = b_i$ on $[0, 1]$.

The univariate example indicates how to proceed in general. The starting point is the the B-spline representation of polynomials. There is a beautiful formula due to Marsden, which we state below in a less precise, qualitative form (cf. e.g. [1] for the univariate case).

[1] Of course, in the univariate case, there is no need to choose a partition which does not include the interval endpoints. However, we would like to mimic the multivariate situation.

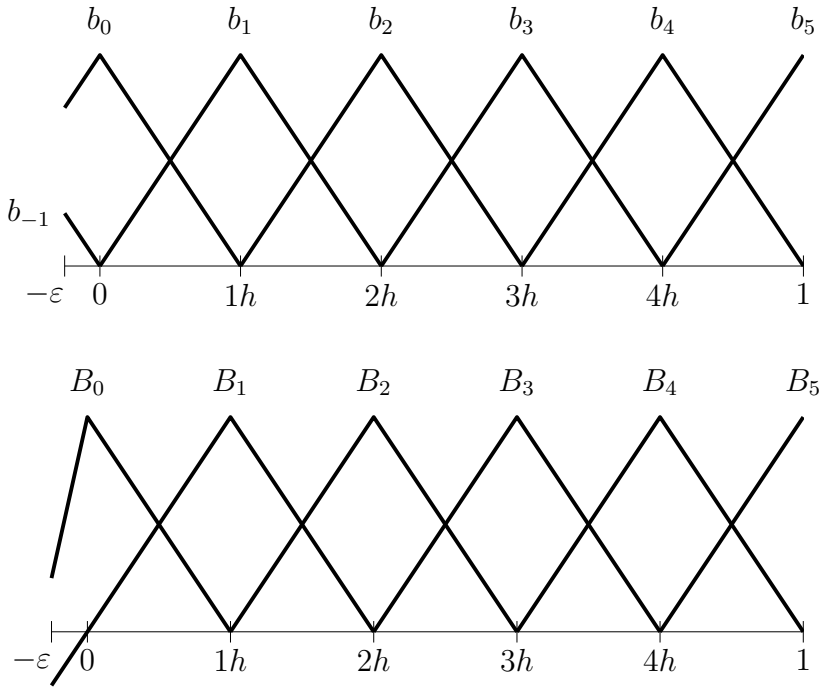

FIGURE 2. Stable basis for broken lines.

Theorem 3. *Any polynomial p of coordinate order n can be uniquely represented as a linear combination*

$$p = \sum_k q(k) b_k, \tag{4}$$

where q is a polynomial of the same coordinate order.

Restricting the above identity to Ω, the sum has to be taken only over

$$k \in K = I \cup J. \tag{5}$$

We now rewrite the terms involving the B-splines b_j, $j \in J$, with small support in Ω. Since q is a polynomial of coordinate degree $\leq n$, we can compute any coefficient $q(j)$, $j \in J$, from n^m coefficients $q(i)$, $i \in I(j)$, if the set of lattice points $I(j)$ is chosen, so that the polynomial interpolation problem is uniquely solvable. We interpolate the values $q(i)$ at the lattice points in $I(j)$ and evaluate the interpolant at j. Hence, if $e_{i,j}$ denotes the value of the Lagrange polynomial associated with $i \in I(j)$ at j, we have

$$q(j) = \sum_{i \in I(j)} e_{i,j} q(i).$$

For simplicity, we set $e_{i,j} = 0$ for $i \notin I(j)$, so we may sum over all $i \in I$. Inserting the expression for $q(j)$ into (4) and interchanging sums gives

$$p(x) = \sum_{i\in I} q(i) \left[b_i(x) + \sum_{j\in J} e_{i,j} b_j(x) \right], \quad x \in \Omega.$$

From this identity we can read off the definition of the basis functions B_i given in (1).

We summarize the above construction in the following Definition, making an explicit choice of the sets $I(j)$, which yields a simple formula for the coefficients $e_{i,j}$.

Definition 1. *For $j \in J$ let $I(j) \subset I$ be an m-dimensional array closest to j with*

$$\alpha_\nu \le i_\nu \le \alpha_\nu + n - 1, \quad \nu = 1 : m,$$

assuming that h is small enough, so that an array of this type exists. Moreover, denote by

$$e_{i,j} = \prod_{\nu} \prod_{\substack{\ell=0:n-1 \\ \alpha_\nu+\ell\neq i_\nu}} \frac{j_\nu - \alpha_\nu - \ell}{i_\nu - \alpha_\nu - \ell} \tag{6}$$

the values of the Lagrange polynomials and set $e_{i,j} = 0$ for $i \notin I(j)$. Then,

$$B_i := b_i + \sum_{j\in J} e_{i,j} b_j, \quad i \in I,$$

is the stabilized B-spline basis.

We note, that the coefficients $e_{i,j}$ are nonzero only if i is close to a boundary index j, i.e.,

$$e_{i,j} \neq 0 \implies \|i - j\| \preceq 1. \tag{7}$$

This implies

$$|e_{i,j}| \preceq 1, \tag{8}$$

by formula (6), and $B_i = b_i$ for B-splines with support sufficiently separated from the boundary.

Figure 3 shows the coefficients $e_{i,j}$, $i \in I(j)$, for two boundary B-splines b_j of order three. They are positioned at the support center of b_i, $i \in I(j)$. As in these examples, the lattice point i usually is a direct neighbor of the array $J(i)$.

Establishing the conditions (**B**) for the new spline basis is very simple. Condition ($\mathbf{B_A}$) is satisfied by construction. Conditions ($\mathbf{B_L}$) and ($\mathbf{B_N}$) follow from (7) and (8). To prove stability, we denote by Ω' the union of all mesh cells Q_k completely contained in Ω. By definition of the sets I and J, all B-splines b_j, $j \in J$, vanish on Ω', while the support of each B-spline b_i, $i \in I$, contains at least one mesh cell from Ω'. Hence, $B_i = b_i$ on Ω' and ($\mathbf{B_S}$) follows from the stability of the B-spline basis for cardinal splines.

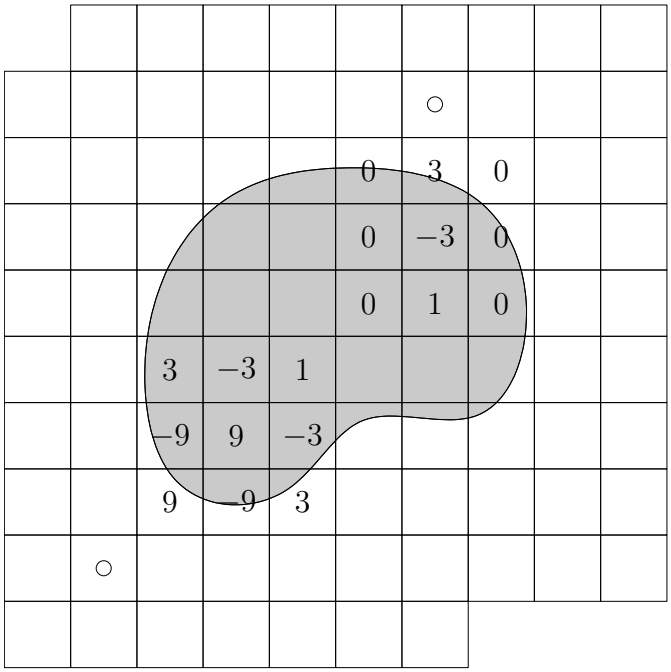

FIGURE 3. Coefficients for combining quadratic B-splines.

4. Implementation and Numerical Results

To illustrate the performance of our method, we compute the flow through a channel with circular obstacles (cf. Figure 4). For an incompressible fluid we can determine the velocity v as the negative gradient of a potential u which satisfies

$$\begin{array}{rcll} \Delta u & = & 0 & \text{in } \Omega \\ \partial^{\perp} u & = & g & \text{on } \partial\Omega. \end{array} \tag{9}$$

In our example, $g = -v_0$ (v_0) on the left (right) end of the channel, and on the other parts of $\partial\Omega$ the normal component of the velocity is zero. Figures 5 and 6 visualize the solution for this model problem by showing the flow speed and the streamlines.

The boundary value problem (9) can be brought into the form (2) by subtracting from u a function, which satisfies the boundary conditions. However, it is simpler to use the direct Galerkin approximation

$$\int_{\Omega} \nabla u_h (\nabla B_i)^t = \int_{\partial\Omega} g B_i, \quad i \in I.$$

In this case, precomputed values can be used for most entries of the Galerkin matrix. Only the integrals over cells intersected by the circular boundaries need to be computed numerically.

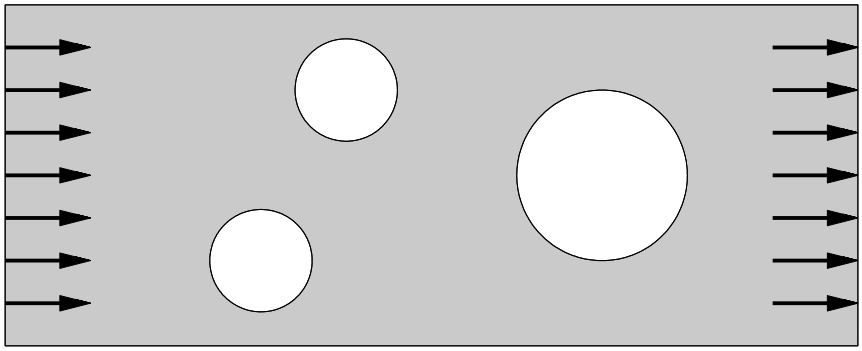

FIGURE 4. Flow through a channel with obstacles

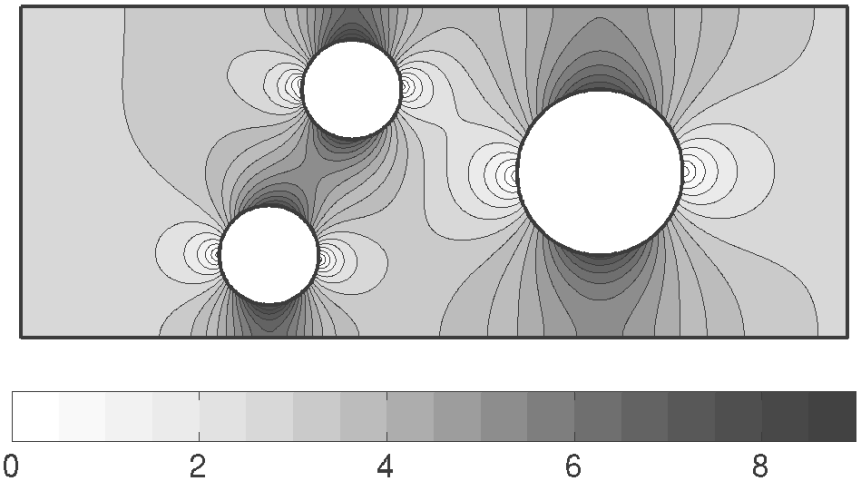

FIGURE 5. Flow speed $\|\nabla u\|$.

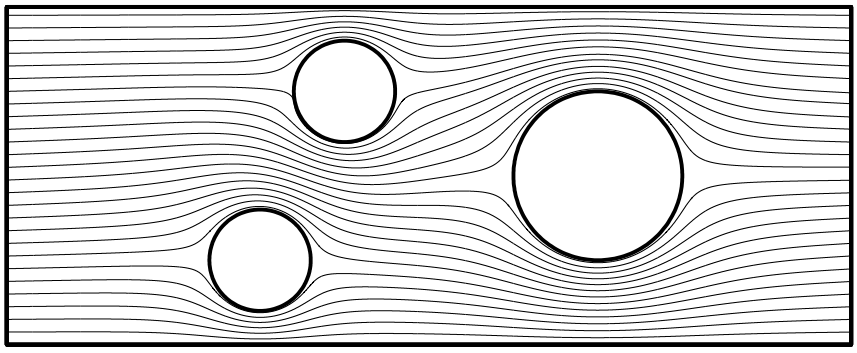

FIGURE 6. Streamlines $\{u = \text{const}\}$.

Figure 7 illustrates the classification of the B-splines. According to (5) there are 44 outer B-splines and 700 inner B-splines, only 202 of which have to be extended.

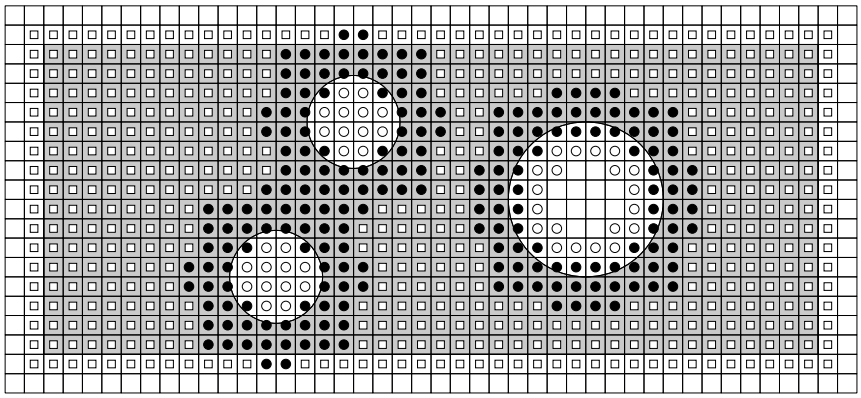

FIGURE 7. Classification of B-Splines: outer (∘), extended inner (•), and unextended inner (□).

Using stabilized B-splines yields a significant increase in accuracy compared to standard methods. This is not only because of the higher degree, but also due to the smoothness of the basis, which reduces the number of parameters.

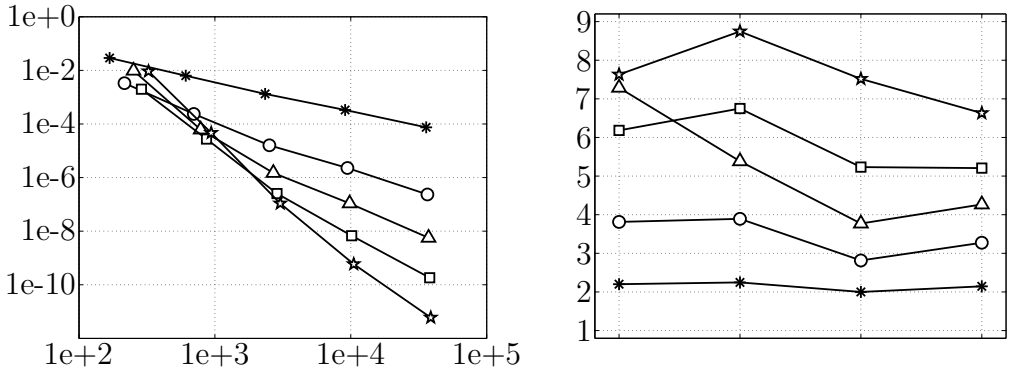

FIGURE 8. Relative L_2 error vs. number of basis functions and rate of convergence.

Figures 8 and 9 show on the left the L^2- and H^1-error versus the number of basis functions. The diagrams on the right confirm the predicted convergence rates, which were estimated by comparing errors when halving the grid width.

We solve the Galerkin system with a standard pcg-iteration. As we already remarked, the fact that the matrix G_h is singular causes no problems. Figure 10 shows that the number of pcg-iterations for a relative accuracy of $1e-11$ does not grow too fast with the dimension. This documents that the stabilization significantly improves the condition of the B-spline basis.

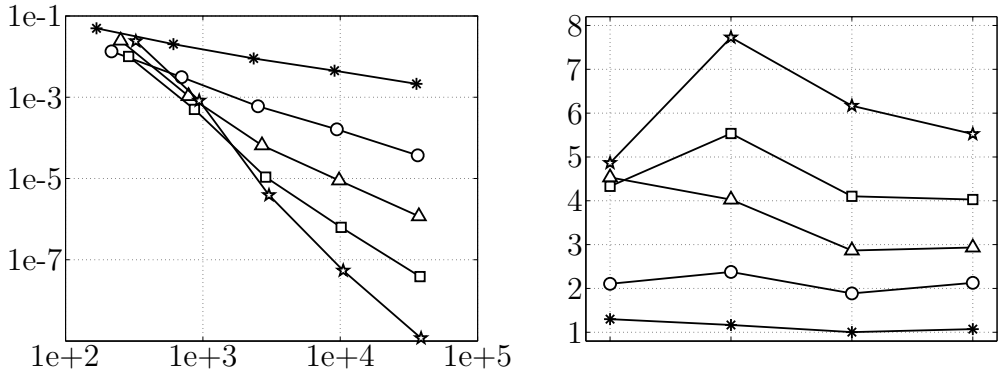

FIGURE 9. Relative H^1 error vs. number of basis functions and rate of convergence.

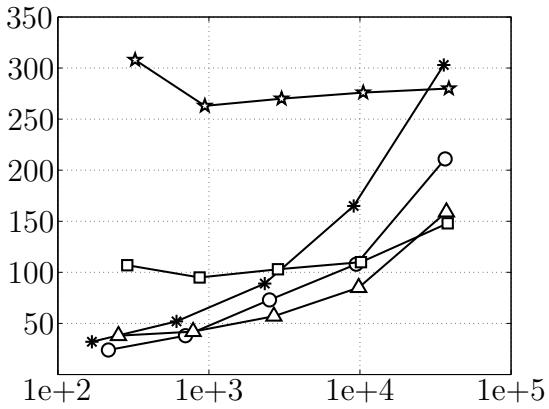

FIGURE 10. Number of pcg iterations vs. number of basis functions.

5. Remarks and Extensions

Our new method applies to fairly general elliptic problems with natural boundary conditions. Moreover, we can derive many of the typical error estimates for finite element approximations and also make the regularity of the data more precise. In conjunction with the penalty method (cf., e.g., [4]) it is also possible to treat other boundary conditions. However, in [6], we describe a much simpler approach. Considering as a model problem Possion's equation with Dirichlet boundary conditions, we use the basis functions

$$\frac{w}{w(x_i)} B_i,$$

where w is a smoothed distance function to the boundary. These weighted B-splines also satisfy conditions (**B**), and thus meet the requirements for standard finite elements. An appropriate combination of both approaches can be applied to most of the standard types of boundary conditions.

The approximation with B-splines on regular grids is ideally suited for hierarchical refinement. Therefore, we expect that multigrid algorithms will yield an even more efficient solution of the linear systems. Moreover, with the multilevel spline spaces constructed in [8] adaptive approximations with wavelet techniques are possible.

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